Nonlinear open mapping principles, with applications to the Jacobian equation and other scale-invariant PDEs

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Abstract

For a nonlinear operator T satisfying certain structural assumptions, we prove that the following statements are equivalent: i) T is surjective, ii) T is open at zero, and iii) T has a bounded right inverse. The result applies to numerous scale-invariant PDEs in regularity regimes where the equations are stable under weak* convergence. As examples we consider the Jacobian equation and the incompressible Euler equations.

For the Jacobian, it is a long standing open problem to decide whether it is onto between the critical Sobolev space and the Hardy space. Towards a negative answer, we show that, if the Jacobian is onto, then it suffices to rule out the existence of surprisingly well-behaved solutions. We also prove that the data-to-solution map is poorly-behaved, by giving explicit examples where there are uncountably many energy-minimal solutions.

For the incompressible Euler equations, we show that, for any $p < \infty$, the set of initial data for which there are dissipative weak solutions in $L^p_t L^2_x$ is meagre in the space of solenoidal L^2 fields.

Contents

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1 Introduction

The open mapping theorem is one of the cornerstones of functional analysis. When X and Z are Banach spaces and $L\colon X \to Z$ is a bounded *linear* operator, it asserts the equivalence of the following two conditions:

(i) Qualitative solvability: for all $f \in Z$ there is $u \in X$ with $Lu = f$, that is, $L(X) = Z$;

(ii) Quantitative solvability: for all $f \in Z$ there is $u \in X$ with $Lu = f$ and $\|u\|_X \le C\|f\|_Z$.

From a PDE perspective, the open mapping theorem justifies the method of a priori estimates [68, §1.7]. For applications to nonlinear PDE, one would like to have an analogue of the open mapping theorem in the case where L is replaced by a *nonlinear* operator $T\colon X \to Z$, and this is the main theme of the present paper.

1.1 A nonlinear open mapping principle

Attempts to adapt the open mapping theorem to nonlinear operators have been spurred by the following problem of Rudin [62, page 67]:

Question 1.1. If X, Y and Z are Banach spaces and T is a continuous bilinear map of $X \times Y$ onto Z, does it follow that T is open at $(0,0)$?

Following [39], we say that *the open mapping principle holds for T* if T is open at the origin. It is easy to see that, in general, T is not open at all points.

In general, the answer to Question 1.1 is negative. Cohen gave a counter-example in [19] and, shortly thereafter, Horowitz gave in [39] a very simple example, consisting of the operator $T\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x,y) \equiv (x_1y_1, x_1y_2, x_1y_3 + x_3y_1 + x_2y_2, x_3y_2 + x_2y_1),$$

see also [8, 29].

Nonetheless, as the main theorem of this paper, we find a natural set of conditions under which the open mapping principle holds:

Theorem A. *Let X and Y be Banach spaces such that $\mathbb{B}_{X^*}$ is sequentially weak* compact. We make the following assumptions:*

(A1) *$T\colon X^* \to Y^*$ is a weak*-to-weak* sequentially continuous operator.*

(A2) *$T(au) = a^sT(u)$ for all $a > 0$ and $u \in X^*$, where $s > 0$.*

(A3) *For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}\colon X^* \to X^*$, $\sigma_k^{Y^*}\colon Y^* \to Y^*$ such that*

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \qquad \text{for all } k \in \mathbb{N}, \qquad \sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \text{for all } f \in Y^*.$$

Then the following conditions are equivalent:

(i) *T is onto: $T(X^*) = Y^*$;*

(ii) *T is open at the origin;*

(iii) *For every $f \in Y^*$ there exists $u \in X^*$ such that $Tu = f$ and $\|u\|_{X^*}^s \le C\|f\|_{Y^*}$.*

Here and in the sequel $\mathbb{B}_{X^*}$ denotes the unit ball of X^*. The hypothesis that $\mathbb{B}_{X^*}$ is sequentially weak* compact holds, for instance, whenever X is separable or reflexive. In general, the weak* topology on a dual space depends on the specific choice of predual [68, Remark 1.9.10] and in fact it is only in this way that the spaces X and Y play a role in the statement of Theorem A.

The assumption (A1) is not always necessary, but it holds automatically in finite dimensional examples. An infinite dimensional case where it is not needed is the multiplication operator $(f,g) \mapsto fg \colon L^p \times L^q \to L^r$, where $1/p + 1/q = 1/r$: this operator does not satisfy (A1), although it verifies the open mapping principle [3, 4].

In Theorem A, instead of considering multilinear operators as in Question 1.1, we consider the larger class of positively homogeneous operators, that is, operators satisfying (A2). Nonetheless, many of the examples discussed in this paper are in fact multilinear.

The assumption (A3) may look somewhat mysterious. However, as HOROWITZ's example shows, it cannot be omitted. It should be thought of as *generalized translation invariance* and indeed, when T is a constant-coefficient partial differential operator and X^* and Y^* are function spaces on $\mathbb{R}^n$, natural choices of $\sigma_k^{X^*}$ and $\sigma_k^{Y^*}$ include translations

$$\sigma_k^X u(x) \equiv u(x-ke) \text{ and } \sigma_k^{Y^*} f(x) \equiv f(x-ke), \qquad \text{where } e \in \mathbb{R}^n \setminus \{0\}.$$

We note that the condition (A3) never holds if Y is finite-dimensional and that moreover, when the target is two-dimensional, it is not needed: DOWNEY has shown that, in this case, the answer to Question 1.1 is positive [30, Theorem 12].

In order for the reader to have a better grasp of the meaning of (A3), we briefly sketch the proof of Theorem A, explaining the role of this assumption in it. Suppose that, for all f in some ball $B \subset Y^*$, one can solve the equation $Tu = f$. Since T is weakly* continuous, it is not difficult to use the Baire Category Theorem to deduce that there is a sub-ball $B' \subseteq B$ such that one can actually solve $Tu = f$ quantitatively in B': that is, there is a constant C such that, for all $f \in B'$, there is $u \in X^*$ with $\|u\|_{X^*} \leq C$ and $Tu = f$. In other words: for weakly* continuous operators, qualitative solvability implies quantitative local solvability *somewhere*; in general, one cannot even specify the location of B'. The assumption (A3) allows one to shift the centre of the ball B' to the origin and, in combination with (A2), it upgrades the previous local statement to a global version.

The simple example $T \colon \mathbb{R} \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), (t,f) \mapsto tf$ shows that there are operators satisfying the assumptions of Theorem A which are not open at all points [30].

Compensated compactness theory [56, 69] abounds with operators that satisfy the assumptions of Theorem A. The most famous examples are the Jacobian, the Hessian and the div-curl product; see [20] for numerous examples and [36, 37] for a systematic study. In fact, our motivation for Theorem A came from considering the Jacobian operator and the spaces

$$X^* = \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n), \qquad Y^* = \mathscr{H}^p(\mathbb{R}^n), \qquad 1 \leq p < \infty, \tag{1.1}$$

see Question 1.2 below. The real-variable Hardy space $\mathscr{H}^p(\mathbb{R}^n)$ is defined by fixing any $\Phi \in \mathscr{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x)\,\mathrm{d}x \neq 0$, denoting $\Phi_t(x) \equiv \Phi(x/t)/t^n$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and setting

$$\mathscr{H}^p(\mathbb{R}^n) \equiv \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\mathscr{H}^p} \equiv \| \sup_{t>0} |f * \Phi_t(\cdot)| \|_{L^p} < \infty \Big\}.$$

We refer the reader to the monograph [66] for the theory of $\mathscr{H}^p(\mathbb{R}^n)$. Here we just note that $\mathscr{H}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, with equivalent norms, whenever $p \in (1,\infty)$ and that moreover

$\mathscr{H}^1(\mathbb{R}^n) \subsetneq \{f \in L^1(\mathbb{R}^n) \colon \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0\}$. Indeed, loosely speaking, elements of $\mathscr{H}^1(\mathbb{R}^n)$ have an extra logarithm of integrability [65]. In (1.1), the space $\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ is the usual homogeneous Sobolev space, seen as a Banach space, so that elements which differ by constants are identified.

A standard way of showing that an operator is open at some point is to verify that its derivative is surjective at that point, that is, that it verifies Lyusternik's condition. However, for $p \in [1,2)$ and for the spaces in (1.1), a Banach space geometry argument shows that neither the Jacobian nor *any other operator can ever verify this condition*, see Section 3. Thus, to study openness of operators between the spaces in (1.1), novel methods are required.

It is possible to refine Theorem A in such a way that it applies in much more general situations. We will return to this point in Section 1.4 below, where we also discuss the consequences of these improvements for certain nonlinear PDEs.

1.2 Applications to the Jacobian equation

As discussed in the last subsection, we are particularly interested in the Jacobian and, therefore, in the prescribed Jacobian equation

$$\mathrm{J}u \equiv \det \mathrm{D}u = f \qquad \text{in } \mathbb{R}^n. \tag{1.2}$$

This first-order equation appears naturally in Optimal Transport [13] and can be seen as the underdetermined analogue of the Monge–Ampère equation, see [34] for further discussion. It has a deep geometric content as, formally, one has the change of variables formula

$$\int_E \mathrm{J}u(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \#\Big(u^{-1}(y) \cap E\Big)\,\mathrm{d}y, \qquad E \subset \mathbb{R}^n \text{ is measurable.} \tag{1.3}$$

Thus, if u is a solution of (1.2), f measures the size of its image, counted with multiplicity.

We consider the following long-standing open problem:

Question 1.2. For $p \in [1,\infty)$ and $f \in \mathscr{H}^p(\mathbb{R}^n)$, is there $u \in \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ solving (1.2)?

To understand the motivation behind Question 1.2, it is important to recall that the Jacobian benefits from an improved integrability: after a remarkable result of Müller [55], Coifman, Lions, Meyers and Semmes proved in [21] that

$$u \in \dot{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n) \quad \Longrightarrow \quad \mathrm{J}u \in \mathscr{H}^1(\mathbb{R}^n).$$

We note that, contrary to what was originally asked in [21], the third author showed in [51] that Question 1.2 must be formulated in terms of the homogeneous Sobolev space, if it is to have a positive answer.

A positive answer to Question 1.2 seems currently out of reach, as there is no systematic way of building solutions to (1.2) for a general discontinuous f, although see [60] for some endpoint cases. This is in contrast with the case of Hölder continuous f, where there is a well-posedness theory which goes back to the works of Dacorogna and Moser [24, 54], see also [23] and the references therein.

In this subsection, we focus on progress towards a negative answer to Question 1.2. The main difficulty in proving non-existence of solutions to (1.2) is the underdetermined nature of the equation: it implies that there is a multitude of possible solutions to rule out. Our investigations towards a negative answer to Question 1.2 are partially motivated from our negative results [34] concerning the following variant of this question: one replaces $\mathbb{R}^n$ with a bounded, smooth domain Ω and one imposes Dirichlet boundary conditions on solutions.

Question 1.2 is much more difficult than its analogue on a bounded domain, as the lack of boundary conditions makes the problem even more underdetermined, see already Theorem D below for a striking illustration of this phenomenon. The space $\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ is both extremely large, and so there is an abundance of possible solutions to consider, and contains many poorly-behaved maps, especially when $p=1$. We now precise this last point.

When $p=1$, there are continuous maps in $\dot{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$ which do not satisfy the change of variables formula (1.3), as they map a null set E into a set of positive measure, and hence

$$0=\int_E f\,\mathrm{d}x<|u(E)|\leq\int_{\mathbb{R}^n}\#\Big(u^{-1}(y)\cap E\Big)\,\mathrm{d}y.$$

It is thus possible for the equation (1.2) to hold a.e. in $\mathbb{R}^n$, and hence also in the sense of distributions, and yet for its geometric information to be completely lost. Maps as above are said to violate the *Lusin (N) property*; their existence is classical and goes back to the work of Cesari [17], see also [52] for a more refined version.

If the answer to Question 1.2 is negative then Theorem A is likely to play a key role in proving so. To see why, let us first explicitly state the following corollary:

Corollary B. *Fix $1\leq p<\infty$. The following statements are equivalent:*

(i) *for all $f\in\mathscr{H}^p(\mathbb{R}^n)$ there is $u\in\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ such that $\mathrm{J}u=f$;*

(ii) *for all $f\in\mathscr{H}^p(\mathbb{R}^n)$ there is $u\in\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ such that $\mathrm{J}u=f$ and*

$$\|\mathrm{D}u\|^n_{L^{np}(\mathbb{R}^n)}\lesssim\|f\|_{\mathscr{H}^p(\mathbb{R}^n)}\tag{1.4}$$

Thus, if the Jacobian $\mathrm{J}\colon\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)\to\mathscr{H}^p(\mathbb{R}^n)$ is surjective, then there are solutions satisfying the a priori estimate (1.4). The crucial point is that, if such an estimate holds, then one may use a regularisation argument, together with powerful machinery from Geometric Function Theory, to prove that there must exist rather well-behaved solutions. We rephrase this loosely in the following principle:

the existence of rough solutions implies the existence of well-behaved solutions.

Hence, to give a negative answer to Question 1.2, it is enough to rule out the existence of well-behaved solutions. The above principle is made precise in the following result:

Theorem C. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and take $f\in\mathscr{H}^1(\mathbb{R}^n)$ such that $f\geq 0$ in Ω. Assume that $\mathrm{J}\colon\dot{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n)\to\mathscr{H}^1(\mathbb{R}^n)$ is onto. Then there is a solution $u\in\dot{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$ of* (1.2) *such that:*

(i) *u is continuous in Ω;*

(ii) *u has the Lusin (N) property in Ω;*

(iii) *$\int_{\mathbb{R}^n}|\mathrm{D}u|^n\,\mathrm{d}x\leq C\|f\|_{\mathscr{H}^1}$ with $C>0$ independent of f.*

In particular, u satisfies the change of variables formula (1.3).

Moreover, if $n=2$ and there is an open set $\Omega'\subseteq\Omega$ with $f=0$ a.e. in Ω', then:

(iv) *for any set* $E \subset \Omega'$, *we have* $u(\partial E) = u(\overline{E})$;

(v) *for* $y \in u(\Omega')$, *if* C *denotes a connected component of* $u^{-1}(y) \cap \Omega'$ *then* C *intersects* $\partial\Omega'$.

In the supercritical regime $p > 1$, the first part of Theorem C holds automatically. Nonetheless, one can still use the a priori estimate (1.4), together with regularisation arguments, to get solutions with additional structure, see Section 5.1 for further details. The second part of Theorem C also holds in any dimension if p is taken to be sufficiently large.

1.3 Other approaches to the Jacobian equation and Iwaniec's programme

In the previous subsection we mostly focused on progress towards a negative answer to Question 1.2. Now we turn to progress and evidence towards a positive answer.

The following result establishes an analogue of the atomic decomposition of $\mathscr{H}^1(\mathbb{R}^n)$, giving a weak factorization on $\mathscr{H}^p(\mathbb{R}^n)$ in the spirit of the classical work of Coifman, Rochberg and Weiss [22]:

Theorem 1.3. *Let* $p \in [1, \infty)$. *For every* $f \in \mathscr{H}^p(\mathbb{R}^n)$ *there are functions* $u_i \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ *and real numbers* c_i *such that*

$$f = \sum_{i=1}^{\infty} c_i \mathrm{J} u_i, \qquad \|u_i\|_{\dot{W}^{1,np}(\mathbb{R}^n)} \leq 1, \quad \sum_{i=1}^{\infty} |c_i| \lesssim \|f\|_{\mathscr{H}^p(\mathbb{R}^n)}. \tag{1.5}$$

In particular, $\mathscr{H}^p(\mathbb{R}^n)$ *is the smallest Banach space containing the range* $\mathrm{J}(\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n))$.

Theorem 1.3 was proved in [21] for $p = 1$, while the case $p > 1$ is much harder and was established only recently by Hytönen in [41]. Question 1.2 can be restated as follows:

Question 1.4. Does the weak factorization (1.5) improve into a *strong factorization*?

In light of the surjectivity result

$$\mathscr{H}^1(\mathbb{R}) = \left\{ \mathcal{H}\omega\, \mathcal{H}\gamma - \omega\, \gamma \colon \omega, \gamma \in L^2(\mathbb{R}) \right\}, \tag{1.6}$$

one may hope for a positive answer to Question 1.4; here $\mathcal{H}$ denotes the Hilbert transform. Indeed, (1.6) is a direct consequence of the strong factorization $\mathscr{H}^1(\mathbb{C}_+) = \mathscr{H}^2(\mathbb{C}_+) \cdot \mathscr{H}^2(\mathbb{C}_+)$ of analytical Hardy spaces and a proof can be found in [49].

To draw a closer analogy between Question 1.4 and (1.6) we note that, in the plane, Question 1.4 is equivalent to deciding whether

$$\mathscr{H}^p(\mathbb{R}^2) = \left\{ |\mathcal{S}\omega|^2 - |\omega|^2 \colon \omega \in L^{2p}(\mathbb{R}^2, \mathbb{R}^2) \right\},$$

where $\mathcal{S}$ is the Beurling–Ahlfors transform. We remark that one may think of $\mathcal{S}$ as the square of a complex Hilbert transform [44]. We may also write

$$\left\{ \mathrm{J}u \colon u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2) \right\} = \left\{ E \cdot B \colon E, B \in L^{2p}(\mathbb{R}^2, \mathbb{R}^2), \operatorname{div} E = 0, \operatorname{curl} B = 0 \text{ in } \mathscr{D}'(\mathbb{R}^2) \right\}.$$

As such, the case $n = 2$ is a fundamental question about the structure of $\mathscr{H}^p(\mathbb{R}^2)$. This viewpoint and the intimate ties to commutators are studied in a general framework in [41, 49].

We now turn towards a less harmonic-analytical approach to Question 1.2. In [43], see also [11], Iwaniec went further than Question 1.2 and conjectured the following:

Conjecture 1.5. For each $p \in [1, \infty)$, the Jacobian has a *fundamental solution*: there is a continuous map $E\colon \mathscr{H}^p(\mathbb{R}^n) \to \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\mathrm{J} \circ E = \mathrm{Id}$.

If $f \in \mathscr{H}^p(\mathbb{R}^n)$ and $\mathrm{J}v = f$ has a solution $v \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$, then there exists a *p-energy minimiser* for f, that is, a solution u with $\int_{\mathbb{R}^n} |\mathrm{D}u|^{np}\,\mathrm{d}x = \min\{\int_{\mathbb{R}^n} |\mathrm{D}v|^{np}\,\mathrm{d}x \colon \mathrm{J}v = f\}$. For a dense set of data, there exists a solution and, consequently, a p-energy minimiser. In [43], Iwaniec proposed the following route to Conjecture 1.5:

Strategy. A possible way of proving Conjecture 1.5 is to establish the following three claims:

(i) Every p-energy minimiser satisfies $\int_{\mathbb{R}^n} |\mathrm{D}u|^{np}\,\mathrm{d}x \lesssim \|\mathrm{J}u\|^p_{\mathscr{H}^p}$.

(ii) Given $f \in \mathscr{H}^p(\mathbb{R}^n)$, the p-energy minimiser for f is unique up to rotations.

(iii) The set-valued map from $f \in \mathscr{H}^p(\mathbb{R}^n)$ to its p-energy minimisers $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ has a continuous selector.

Corollary B says that (i) is equivalent to a positive answer to Question 1.2. In this direction, Iwaniec has suggested that one should prove (i) by constructing a *Lagrange multiplier* for every p-energy minimiser. However, standard methods fail, see Section 3 for further discussion. Nevertheless, in the case $n = 2$, $p = 1$, a Lagrange multiplier is constructed in [49, 50] for a large class of p-energy minimisers, which then automatically satisfy the a priori estimate $\int_{\mathbb{R}^2} |\mathrm{D}u|^2 \lesssim \|\mathrm{J}u\|_{\mathscr{H}^1}$. The methods of [50, 49] can be partly adapted to all the cases $n \geq 2$, $p \in [1, \infty)$.

Our main contribution in relation to Iwaniec's programme is to show that (ii) is false:

Theorem D. *Fix $1 \leq p < \infty$. There is a radially symmetric function $f \in \mathscr{H}^p(\mathbb{R}^n)$ which has uncountably many p-energy minimisers, modulo rotations.*

Theorem D shows that (1.2) is tremendously underdetermined. It also seems difficult to work directly with p-energy minimisers: for instance, when $p = 1$, we cannot decide whether they are necessarily continuous, not even over open sets where $f \geq 0$. One possible takeaway from Theorem D is that p-energy minimisers may not be the right solution to consider and that instead one should study the potentially more regular solutions obtained from Theorem C.

To conclude the discussion of the above strategy, we note that claim (iii) is a nonlinear analogue of the classical Bartle–Graves theorem [7], see [9, page 86] for a good overview. This theorem says that a bounded linear surjection between Banach spaces has a (possibly nonlinear) continuous right inverse. Without extra assumptions, the Bartle–Graves theorem does not generalise to multilinear mappings: in [32] it was constructed, for every $m \geq 2$, a continuous m-linear surjection $T\colon X_1 \times \cdots \times X_m \to Z$ between Banach spaces which is open at the origin but has no continuous right inverse. However, as a partial result towards (iii), assuming surjectivity of $\mathrm{J}\colon \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$, one may use Corollary B to find a bounded right inverse that is continuous outside a meagre set, although we do not prove such a result here.

1.4 More general nonlinear open mapping principles and applications to other PDEs

In Theorem 6.2 the positive homogeneity assumption of Theorem A is replaced by more general scaling symmetries. For motivation, note that the positive n-homogeneity of the

Jacobian operator can be expressed as symmetry of the equation $\mathrm{J}u = f$ under the scaling $u_\lambda = \lambda u$, $f_\lambda = \lambda^n f$ for all $\lambda > 0$ but the Jacobian equation also has the other scaling symmetry $u_\lambda(x) = \lambda u(x/\lambda)$, $f_\lambda(x) = f(x/\lambda)$.

We describe Theorem 6.2 in non-technical terms. The main applications are to nonlinear constant-coefficient PDE's $Tu = f$, where u and f belong to function spaces X^* and Y^* on $\mathbb{R}^n$ or, in the case of evolutionary problems, on $\mathbb{R}^n \times [0, \infty)$. As before, T is assumed to be weak*-to-weak* continuous (and translation invariant).

We assume that the equation $Tu(x,t) = f(x,t)$, $(x,t) \in \mathbb{R}^n \times [0,\infty)$, is invariant under a one-parameter group of scalings

$$u_\lambda(x,t) = \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma}\right), \qquad f_\lambda(x,t) = \frac{1}{\lambda^\delta} f\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma}\right),$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are fixed and the group parameter $\lambda > 0$. We mostly focus on *homogeneous* function spaces which satisfy

$$\|u_\lambda\|_{X^*} \equiv \lambda^r \|u\|_{X^*}, \qquad \|f_\lambda\|_{Y^*} \equiv \lambda^s \|f\|_{Y^*}$$

for some $r, s \in \mathbb{R}$. Assuming that either $r, s > 0$ or $r, s < 0$, Theorem 6.2 states the equivalence of the following two claims (the critical case $r = s = 0$ is beyond the scope of this work):

(i) For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$;

(ii) For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$ and $\|u\|_{X^*}^{s/r} \leq C\|f\|_{Y^*}$.

This provides a far-reaching generalisation of the Banach–Schauder open mapping theorem which applies to numerous equations arising from physics, at least in regularity regimes where the equations are stable under weak* convergence.

Indeed, invariance under translations and scalings is a ubiquitous feature of natural processes; it expresses the *covariance principle* that the solutions of a PDE representing a physical phenomenon should not have a form which depends on the location of the observer or the units that the observer is using to measure the system [16]. For the computation of the symmetry groups of several representative PDEs from physics we refer to [57, §2.4] and for the general role of scaling symmetries in physics and other sciences to [6].

We illustrate the applications of Theorem 6.2 by considering energy dissipating solutions of the incompressible Euler equations

$$\partial_t u + u \cdot \nabla u - \nabla P = 0, \tag{1.7}$$

$$\operatorname{div} u = 0, \tag{1.8}$$

$$u(\cdot, 0) = u^0 \tag{1.9}$$

in $\mathbb{R}^n \times [0, \infty)$, $n \geq 2$. Note that (1.7)–(1.9) are invariant under all the scalings of the form

$$u_\lambda(x,t) \equiv \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right), \quad u^0_\lambda(x,t) \equiv \frac{1}{\lambda^\alpha} u^0\left(\frac{x}{\lambda^\beta}\right), \quad P_\lambda(x,t) \equiv \frac{1}{\lambda^{2\alpha}} P\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right).$$

Solutions which fail to conserve energy have been studied extensively in relation to the so-called *Onsager conjecture*; see [15, 26, 42] and the references therein. By a theorem of Székelyhidi and Wiedemann [67], for a dense set of data $u^0 \in L^2_\sigma$ there exist infinitely many *admissible* solutions $u \in L^\infty_t L^2_{\sigma,x}$ of (1.7)–(1.9), that is, ones satisfying the energy inequality $\int_{\mathbb{R}^n} |u(x,t)|^2 \,\mathrm{d}x \le \int_{\mathbb{R}^n} |u^0(x)|^2 \,\mathrm{d}x$ for all $t \ge 0$. For some data $u^0 \in L^2_\sigma$, a solution can even be chosen to be compactly supported in time; indeed, Scheffer had already constructed in [63] solutions of the Euler equations which are compactly supported and square integrable in space-time. Nevertheless, Theorem 6.2 can be used to show that for a Baire-generic datum $u^0 \in L^2_\sigma$ the kinetic energy $\frac{1}{2}\int_{\mathbb{R}^n} |u(x,t)|^2 \,\mathrm{d}x$ cannot undergo an L^q-type decay, $q < \infty$:

Theorem E. *Let $n \ge 2$ and $2 < p < \infty$. For a residual set of data $u^0 \in L^2_\sigma$ the Cauchy problem* (1.7)–(1.9) *has no solution $u \in L^p_t L^2_{\sigma,x}$. More precisely, for every $M > 0$, the set of data with a solution $u \in M\mathbb{B}_{L^p_t L^2_{\sigma,x}}$ is nowhere dense in L^2_σ.*

Corollary F. *Suppose $\tau > 0$. An admissible solution $u \in L^\infty_t L^2_{\sigma,x}$ with* $\operatorname{supp}(u) \subset \mathbb{R}^n \times [0,\tau]$ *exists only for a nowhere dense set of data $u^0 \in L^2_\sigma$.*

We define the operator T as a map from a solution u of (1.7)–(1.9) to the initial data u^0, thereby turning the Cauchy problem into a question about surjectivity of a nonlinear operator. In fact, in order to make T weak*-to-weak* sequentially continuous, we relax the equation (1.7) into a linear one in the proof of Theorem E.

2 Notation and preliminary results

We write Ω for a domain in $\mathbb{R}^n$. We use polar coordinates $z = re^{i\theta} = x + iy \in \mathbb{C}$ in the plane. We write $B_r(x)$ for the usual Euclidean balls in $\mathbb{R}^n$, and $S_r \equiv B_r$ (when x is omitted, it is understood that $x = 0$). It is also useful to have notation for annuli: for $r < R$, $\mathbb{A}(r,R) \equiv \{x \in \mathbb{R}^n : r < |x| < R\}$. Here $|\cdot|$ denotes the Euclidean norm of a vector in $\mathbb{R}^n$ and likewise the Euclidean norm of a matrix $A \in \mathbb{R}^{n\times n}$. Unless stated otherwise, p is in $[1,+\infty)$. The symbols $a \approx b$ and $a \lesssim b$ mean that there is some constant $C > 0$ independent of a and b such that $C^{-1}a \le b \le Ca$ and $a \le Cb$, respectively.

The rest of this section collects, for the convenience of the reader, some useful results about Sobolev functions and Geometric Function Theory.

2.1 Radial stretchings and their generalisations

Given a planar Sobolev map $u \in W^{1,p}(\mathbb{R}^2,\mathbb{R}^2)$, we consider polar coordinates both in the domain and in the target; that is, we want to write

$$u(re^{i\theta}) = \psi(r,\theta)\exp(i\gamma(r,\theta)) \tag{2.1}$$

for some functions $\psi\colon (0,\infty)\times[0,2\pi] \to [0,\infty)$ and $\gamma\colon (0,\infty)\times[0,2\pi] \to \mathbb{R}$, where furthermore we must have the compatibility conditions

$$\psi(r,0) = \psi(r,2\pi) \quad \text{and} \quad \gamma(r,0) - \gamma(r,2\pi) \in 2\pi\mathbb{Z} \qquad \text{for all } r.$$

We will freely identify $(r,\theta) \equiv re^{i\theta}$, adopting either notation whenever it is more convenient.

The existence of a representation as in (2.1) is a standard problem in lifting theory:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth, simply connected domain and $p \ge 2$. Let $u \in W^{1,p}(\Omega,\mathbb{R}^2)$ be such that $0 \notin u(\Omega)$ and, if $p = 2$, suppose moreover that u is continuous.*

There are functions $\psi \in W^{1,p}(\Omega,(0,\infty))$ and $\gamma \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R})$ such that the representation (2.1) holds. If $p=2$ then ψ and γ are also continuous.

Moreover, if $0 \in \Omega$ and $u^{-1}(\{0\}) = \{0\}$, then (2.1) still holds with $\psi \in W^{1,p}(\Omega,[0,\infty))$, $\gamma \in W^{1,p}_{\mathrm{loc}}(\Omega \setminus \{0\},\mathbb{R})$. If $p=2$, then ψ is continuous in Ω and γ is continuous in $\Omega \setminus \{0\}$.

Proof. Note that if $u \in W^{1,p}$ then $\psi = |u|$ is also in $W^{1,p}$. Thus, as $0 \notin u(\Omega)$, it suffices to prove the existence of $\gamma \in W^{1,p}(\Omega,\mathbb{R})$ such that $u/|u| = e^{i\gamma}$. Since u is continuous, $u/|u| \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{S}^1)$, and so the existence of γ follows from the results in [10], see also [12].

For the last part let $\varepsilon > 0$ and consider the keyhole domains

$$\Omega_{1,\varepsilon} = B \setminus \left(\{(r,\theta)\colon |\theta| \le \varepsilon\} \cup B_\varepsilon\right) \qquad \Omega_{2,\varepsilon} = B \setminus \left(\{(r,\theta)\colon |\pi-\theta| \le \varepsilon\} \cup B_\varepsilon\right).$$

From the first part we know that we can write, for $i=1,2$, $u(r,\theta) = \psi(r,\theta)e^{i\gamma_i(r,\theta)}$ in $\Omega_{i,\varepsilon}$ with $\gamma_i \in W^{1,p}(\Omega_{i,\varepsilon})$. For almost every $(r,\theta) \in \Omega_{1,\varepsilon} \cap \Omega_{2,\varepsilon}$,

$$\psi(r)e^{i\gamma_1(r,\theta)} = u = \psi(r)e^{i\gamma_2(r,\theta)} \iff \gamma_1(r,\theta) - \gamma_2(r,\theta) = 2\pi k(r)$$

where $k(r) \in \mathbb{Z}$. As γ_1, γ_2 are continuous in $\Omega_{1,\varepsilon} \cap \Omega_{2,\varepsilon}$, we must have $k(r) = k$, so that without loss of generality, upon redefining, γ_1 we may assume $k=0$. Hence we may define

$$\gamma_\varepsilon(r,\theta) = \begin{cases} \gamma_1(r,\theta) \text{ if } (r,\theta) \in \Omega_{1,\varepsilon} \\ \gamma_2(r,\theta) \text{ if } (r,\theta) \in \Omega_{2,\varepsilon} \end{cases}$$

to find that we may write $u = \psi(r)e^{i\gamma_\varepsilon(r,\theta)}$ with $\gamma_\varepsilon \in W^{1,2}(B \setminus B_\varepsilon)$. By a similar argument, we see that we may take $\gamma_\varepsilon = \gamma_\delta$ in $B \setminus (B_\delta \cup B_\varepsilon)$, so that in fact $u = \psi(r)e^{i\gamma(r,\theta)}$ with $\gamma \in W^{1,2}_{\mathrm{loc}}(B \setminus \{0\})$. Since $u(0) = 0$, ψ extends to a continuous function on Ω. It is now immediate from our construction that the claimed regularity properties hold. □

Remark 2.2. The conclusion of Proposition 2.1 is false if $p < 2$, see [12, §4].

A function $f\colon B_R(0) \to \mathbb{R}$ is said to be *radially symmetric* if $|x| = |y| \implies f(x) = f(y)$ and we identify any such function with a function $f\colon [0,+\infty) \to \mathbb{R}$ in the obvious way. For such a function, it is natural to look for solutions of (1.2) possessing some symmetry, in this case $\partial_\theta\psi = 0$ if a representation as in (2.1) holds. We thus arrive at the following definition:

Definition 2.3. When $p \in [1,\infty)$, $k \in \mathbb{Z}$ and $f \in \mathscr{H}^p(\mathbb{R}^2)$ is radially symmetric, a mapping of the form

$$\phi_k(z) = \phi_k(re^{i\theta}) \equiv \frac{\rho(r)}{\sqrt{|k|}}e^{2\pi ik\theta}, \qquad \rho^2(r) = \int_0^r 2sf(s)\,\mathrm{d}s$$

is called a *generalised radial stretching*. We simply refer to ϕ_1 as a *radial stretching*.

Generalised radial stretchings are also spherically symmetric in the sense that they map circles centred at zero to circles centred at zero.

Clearly it is not the case that any radially symmetric $f \in \mathscr{H}^p(\mathbb{R}^2)$ admits generalised radial stretchings as solutions of (1.2): for instance, we must have

$$\text{either } \int_0^r 2sf(s)\,\mathrm{d}s \le 0 \text{ for a.e. } r, \qquad \text{or } \int_0^r 2sf(s)\,\mathrm{d}s \ge 0 \text{ for a.e. } r.$$

In general, it is not completely clear what the relation between the regularity of f and the regularity of ϕ_k is, but see [34], [50, §3] and [70, §7]. In this direction, the following is a useful criterion:

Lemma 2.4. *Let $k \in \mathbb{Z}$ and ϕ_k be as in Definition 2.3. For $1 \leq p < \infty$, $\phi_k \in \dot{W}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ if and only if ρ is absolutely continuous on $(0, +\infty)$ and*

$$\|\mathrm{D}\phi_k\|^p_{L^p(\mathbb{R}^2)} \approx \int_0^\infty \left(\left| \frac{\dot{\rho}(r)}{k} \right|^p + \left| k\, \frac{\rho(r)}{r} \right|^p \right) r \,\mathrm{d}r < \infty.$$

We omit the proof, as it is a straightforward adaptation of [5, Lemma 4.1].

2.2 The Lusin (N) property and the change of variables formula

The following notions are very relevant in relation to the change of variables formula:

Definition 2.5. Let $u \colon \Omega \to \mathbb{R}^n$ be a continuous map which is differentiable a.e. in Ω. Then:

(i) u has the *Lusin (N) property* if $|u(E)| = 0$ for any $E \subset \Omega$ such that $|E| = 0$;

(ii) u has the *(SA) property* if $|u(E)| = 0$ for any open set $E \subset \Omega$ with $\mathrm{J}u = 0$ a.e. in E.

In the one-dimensional case, the Lusin (N) property is well understood: for instance, on an interval, a continuous function of bounded variation has the Lusin (N) property if and only if it is absolutely continuous. However, in higher dimensions, the situation is much more complicated, although we have the following characterisation, proved in [53]:

Proposition 2.6. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map with $\mathrm{J}u \geq 0$ in Ω. Then u has the Lusin (N) property if and only if it has the (SA) property.*

We remark that Proposition 2.6 is in general false if $\mathrm{J}u \not\geq 0$, see [59] for a counterexample. The following result, see [52], is also useful for our purposes:

Proposition 2.7. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map such that, for some $K \geq 1$,*

$$\operatorname{diam}(u(B_r(x)) \leq K \operatorname{diam}(u(\partial B_r(x))) \qquad \text{for all } B_r(x) \Subset \Omega. \tag{2.2}$$

Then u has the Lusin (N) property.

The change of variables formula is closely related to the Jacobian determinant; the following result, together with the definition of the topological degree, can be found in [33].

Theorem 2.8. *Let $u \in C^0(\Omega, \mathbb{R}^n) \cap W^{1,n}(\Omega, \mathbb{R}^n)$ be a map with the Lusin (N) property. Then*

$$\int_E \mathrm{J}u \,\mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) \,\mathrm{d}y \qquad \text{for all measurable sets } E \subset \Omega, \tag{2.3}$$

where $\mathcal{N}$ is the multiplicity function*, defined as $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$, and*

$$\int_E \mathrm{J}u \,\mathrm{d}x = \int_{\mathbb{R}^n} \deg(y, u, E) \,\mathrm{d}y \qquad \text{for all open sets } E \subset \Omega, \tag{2.4}$$

where $\deg(y, u, E)$ *denotes the* topological degree *of u at y with respect to E.*

2.3 Mappings of finite distortion

In this subsection we recall some useful facts about mappings of finite distortion and, for simplicity, we focus on the planar case $n = 2$, see [2]. The reader can also find these and higher-dimensional results in [38, 44].

Definition 2.9. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be such that $0 \le \mathrm{J}u \in L^1_{\mathrm{loc}}(\Omega)$. We say that u is a *map of finite distortion* if there is a function $K \colon \Omega \to [2, \infty]$ such that $K < \infty$ a.e. in Ω and

$$|\mathrm{D}u(x)|^2 \le K(x)\,\mathrm{J}u(x) \quad \text{ for a.e. } x \text{ in } \Omega.$$

If u has finite distortion, we can set $\mathrm{K}u(x) = \frac{|\mathrm{D}u|^2}{\mathrm{J}u(x)}$ if $\mathrm{J}u(x) \neq 0$ and $\mathrm{K}u(x) = 2$ otherwise; this function is the (optimal) *distortion* of u.

We summarise here some of the key analytic and topological properties of mappings of finite distortion in the plane:

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^2$ and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a map of finite distortion. Then:*

(i) *u has a continuous representative and, whenever $r < R$ and $B_R(x_0) \subset \Omega$,*

$$\left(\operatorname{osc}_{B_r(x_0)} u\right)^2 \le \frac{C}{\log(R/r)} \int_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x;$$

(ii) *u has the Lusin (N) property;*

(iii) *u is differentiable a.e. in Ω;*

(iv) *if $\mathrm{K}u \in L^1(\Omega)$ then u is open and discrete;*

(v) *if $\mathrm{K}u \in L^1(\Omega)$ then for each $\Omega' \Subset \Omega$ there is $m = m(\Omega')$ such that*

$$\mathcal{N}(y, u, \Omega') \le m \qquad \text{for all } y \in u(\Omega').$$

Whenever u is a map of finite distortion we always implicitly assume that u denotes the continuous representative of the equivalence class in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. If u is such that $\mathrm{K}u \in L^1(\Omega)$, we say that u has *integrable distortion*; the theory of such maps was pioneered in [45].

We remark that the first three properties of Theorem 2.10 are a consequence of the fact that mappings of finite distortion are *monotone in the sense of Lebesgue*:

Proposition 2.11. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a map of finite distortion; then* (2.2) *holds. In fact, if we measure the diameter in $\mathbb{R}^n$ with respect to the ℓ^∞ norm, we can take $K = 1$.*

3 The Jacobian is a submersion nowhere

Let X, Y be Banach spaces. We use the following terminology:

Definition 3.1. A function $F \colon X \to Y$ is said to be a *submersion at $x_0 \in X$* if F is Gâteaux-differentiable at x_0 and $F'(x_0) \colon X \to Y$ is onto.

In analogy to the finite dimensional case, if F is a submersion at x_0, it is open at x_0, see for instance [27, Corollary 15.2]. More precisely:

Theorem 3.2. *Let $F\colon X \to Y$ be a locally Lipschitz submersion at $x_0 \in X$. For all $R > 0$ sufficiently small, there is $r > 0$ such that $B_r(F(x_0)) \subseteq F(B_R(x_0))$.*

The submersion condition also plays an important role in Lyusternik's theory of constrained variational problems, through the existence of Lagrange multipliers. We remark that, in that setting, it is customary to additionally require $\ker F'(x_0)$ to be complemented in X. Here we do not discuss further the existence of Lagrange multipliers nor their properties, referring instead the interested reader to [71, §43] for their general theory. In the context of Question 1.2, Lagrange multipliers were considered in the third author's doctoral thesis [50].

The purpose of this section is to concisely illustrate some of the advantages of Theorem A over the very classical Theorem 3.2. In particular, in Proposition 3.4, we show that Theorem 3.2 does not apply to the Jacobian. We begin with the following straightforward lemma:

Lemma 3.3. *Suppose $F\colon X \to Y$ is Gâteaux-differentiable. If Y^* does not embed into X^* then F is a submersion at no point.*

Proof. We prove the contrapositive. Suppose F is a submersion at some $x_0 \in X$, that is, $T \equiv F'(x_0)\colon X \to Y$ is onto. By the classical open mapping principle, $T^*\colon Y^* \to X^*$ is bounded by below and is thus an isomorphism onto its image. Thus Y^* embeds into X^*. □

The main result of this section is the following:

Proposition 3.4. *Let $p \in [1, 2)$ and suppose $T\colon \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ is Gâteaux-differentiable. Then T is a submersion at no point.*

Proof. The case $p = 1$ is simple: $(\mathscr{H}^1(\mathbb{R}^n))^* = \mathrm{BMO}(\mathbb{R}^n)$ is not reflexive and thus it cannot embed into a reflexive space, such as $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)^*$.

For $p \in (1, 2)$, we begin by using the isomorphism $(-\Delta)^{1/2}\colon \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to L^{np}(\mathbb{R}^n, \mathbb{R}^n)$. Thus it suffices to show that $L^{p'}$ does not embed into $L^{(np)'}$ for $p \in (1, 2)$, where q' denotes the Hölder conjugate of q. Since $p' > 2$, we appeal to Lemma 3.5 below to complete the proof. □

Thus, it remains to prove the next lemma, where H is a Hilbert space.

Lemma 3.5. *Let $p \in [1, 2], q \in [1, \infty)$. If $L^q(\mathbb{R}^n)$ embeds into $L^p(\mathbb{R}^n, H)$ then $1 \le p \le q \le 2$.*

This result is well-known to the experts and a very complete statement can be found in [1, Proposition 12.1.10], which we quote here:

Proposition 3.6. *Let $p, q \in [1, \infty)$. Then $L^q(\mathbb{R}^n)$ embeds into $L^p(\mathbb{R}^n)$ if and only if one of the following conditions holds:*

(i) *$1 \le p \le q \le 2$,*

(ii) *$2 < p < \infty$ and $q \in \{2, p\}$.*

Lemma 3.5 is essentially deduced from Proposition 3.6, as the vector-valued L^p space poses only minor changes to the proof. We sketch a proof of Lemma 3.5 here, in order to improve the readability of the paper. The proof relies on the notions of (Rademacher) type and cotype of a Banach space:

Definition 3.7. Let $(\varepsilon_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}.$$

A Banach space X has *type* p, $p \in [1,2]$ if there is a constant C such that

$$\left(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^p\right)^{1/p} \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$$

for any vectors $x_i \in X$. Likewise, X has *cotype* q, $q \in [2,+\infty]$, if there is C such that

$$\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leq C\left(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^q\right)^{1/q}$$

for any vectors $x_i \in X$.

The range of p and q in the definitions of type and cotype are natural and are determined by Khintchine's inequality. Moreover, if X is of type p then it is also of type r for any $r < p$; if it is of cotype q, it is also of cotype r for any $r > q$.

Example 3.8. As before, X is a Banach space.

(i) A Hilbert space H has type and cotype 2: this follows from the parallelogram law.

(ii) If X has type p then X^* has cotype p', although the converse is not true.

(iii) If $p \in [1,2]$ then ℓ^p has type p and if $p \in [2,+\infty]$ then ℓ^p has cotype p. Moreover, these values are optimal, as can be seen by considering the standard basis.

(iv) If X has type p and cotype q, the space $L^r(\mathbb{R}^n, X)$ has type $\min\{r,p\}$ and cotype $\max\{r,q\}$.

The reader may find details and further examples in [1, 40].

Proof of Lemma 3.5. Clearly type and cotype are inherited by subspaces. Thus, if $p \in [1,2]$ and if $L^q(\mathbb{R}^n)$ embeds into $L^p(\mathbb{R}^n, H)$, then $L^q(\mathbb{R}^n)$ must have type p and cotype 2. Since ℓ^q embeds into $L^q(\mathbb{R}^n)$, the same can be said for ℓ^q. Hence, the optimality in Example 3.8(iii) shows that $p \leq q \leq 2$. □

Remark 3.9. Inspection of the proof reveals that, in Proposition 3.4, the following stronger conclusion holds: for any $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$, $T_u' \colon \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ does not have closed range. This condition also appears naturally in relation to the existence of Lagrange multipliers, see e.g. [27, §26.2].

4 A nonlinear open mapping principle for positively homogeneous operators

The main goal of this section is to prove Theorem A. A related nonlinear uniform boundedness principle is proved in Proposition 4.3 and a precise statement concerning atomic decompositions in terms of T is proved in Proposition 4.4.

In the case of the Jacobian, by adapting a standard proof of the standard Open Mapping Theorem to Question 1.2 one obtains the following statement: if $\mathrm{J}(\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)) = \mathscr{H}^p(\mathbb{R}^n)$, then for every $f \in \mathscr{H}^p(\mathbb{R}^n)$ there exist $u, v \in \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ with

$$\mathrm{J}u + \mathrm{J}v = f \qquad \text{and} \qquad \int_{\mathbb{R}^n} (|\mathrm{D}u|^{np} + |\mathrm{D}v|^{np})\, \mathrm{d}x \le C\|f\|^p_{\mathscr{H}^p}.$$

Thus, quantitative control is gained at the expense of introducing an extra term $\mathrm{J}v$. In Theorem A and Corollary B, the extra term is removed, leading to a genuinely nonlinear version of the Open Mapping Theorem.

4.1 The proof of Theorem A

Here we give a slightly more precise version of Theorem A:

Theorem 4.1. *Let X and Y be Banach spaces such that $\mathbb{B}_{X^*}$ is sequentially weak* compact. We make the following assumptions:*

(A1) *$T\colon X^* \to Y^*$ is a weak*-to-weak* sequentially continuous operator.*

(A2) *$T(au) = a^s T(u)$ for all $a > 0$ and $u \in X^*$, where $s > 0$.*

(A3) *For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}\colon X^* \to X^*$, $\sigma_k^{Y^*}\colon Y^* \to Y^*$ such that*

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \qquad \text{for all } k \in \mathbb{N}, \qquad \sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \text{for all } f \in Y^*.$$

Then the following conditions are equivalent:

(i) *$T(X^*)$ is non-meagre in Y^*.*

(ii) *$T(X^*) = Y^*$.*

(iii) *T is open at the origin.*

(iv) *For every $f \in Y^*$ there exists $u \in X^*$ such that*

$$Tu = f, \qquad \|u\|^s_{X^*} \le C\|f\|_{Y^*}. \tag{4.1}$$

A sufficient condition for $\mathbb{B}_{X^*}$ to be sequentially weak* compact is that X is a *weak Asplund space* [64, Theorem 3.5]. For instance, reflexive or separable spaces are weak Asplund [28].

Proof of Theorem A. We have (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and so we just prove (i) $\Rightarrow$ (iv).

Assume that (i) holds. We may write $T(X^*)$ as a union $\cup_{\ell=1}^\infty K_\ell$, where

$$K_\ell \equiv \Big\{ f \in Y^* : \text{there exists } u \in X^* \text{ with } Tu = f \text{ and } \|u\|^s_{X^*} \le \ell \|f\|_{Y^*} \Big\}.$$

Since balls in X^* are sequentially weak* compact, by (A1), the sets K_ℓ are norm-closed. Now, by the Baire Category Theorem, some K_ℓ contains a closed ball $\bar{B}_r(f_0)$.

Our aim is to solve (4.1) whenever $\|f\|_{Y^*} = r$; assumption (A2) then implies the claim. Suppose, therefore, that $\|f\|_{Y^*} = r$. For every $k \in \mathbb{N}$ we have $f_0 + (\sigma_k^{Y^*})^{-1} f \in \bar{B}_r(f_0)$. Hence, we may choose $u_k \in X^*$ such that $Tu_k = f_0 + (\sigma_k^{Y^*})^{-1} f$ and

$$\|\sigma_k^{X^*} u_k\|^s_{X^*} = \|u_k\|^s_{X^*} \le \ell \|f_0 + (\sigma_k^{Y^*})^{-1} f\|_{Y^*} \le \ell(\|f_0\|_{Y^*} + r).$$

Since balls in X^* are sequentially weak* compact, after passing to a subsequence if need be, $\sigma_k^{X^*} u_k$ converges weakly* to some $u \in X^*$, so that $T(\sigma_k^{X^*} u_k) \overset{*}{\rightharpoonup} Tu$. By the lower semicontinuity of the norm we have

$$\|u\|_{X^*}^s \leq \liminf_{k\to\infty} \|\sigma_k^{X^*} u_k\|_{X^*}^s \leq \ell(\|f_0\|_{Y^*} + r).$$

On the other hand, (A3) gives

$$T(\sigma_k^{X^*} u_k) = \sigma_k^{Y^*}(Tu_k) = \sigma_k^{Y^*} f_0 + f \overset{*}{\rightharpoonup} f,$$

so that, by (A1), $Tu = f$. Thus u solves (4.1) and the proof is complete. □

The theory of Compensated Compactness provides many examples of nonlinear operators to which Theorem A applies. Here we give a general formulation in the spirit of [35], see also [56, 69], which we then illustrate with more concrete examples.

Example 4.2. Let $\mathcal{A}$ be an l-th order homogeneous linear operator, which for simplicity we assume to have constant coefficients; that is, for $v \in C^\infty(\mathbb{R}^n, \mathbb{V})$,

$$\mathcal{A}v = \sum_{|\alpha|=l} A_\alpha \partial^\alpha v, \qquad A_\alpha \in \operatorname{Lin}(\mathbb{V}, \mathbb{W}),$$

where $\mathbb{V}, \mathbb{W}$ are finite-dimensional vector spaces. For $p \in [1, +\infty)$ and $s \in \mathbb{N}$, $s \geq 2$, take

$$X^* = L^{ps}_{\mathcal{A}}(\mathbb{R}^n, \mathbb{V}), \qquad Y^* = \mathscr{H}^p(\mathbb{R}^n).$$

Here $L^{ps}_{\mathcal{A}}(\mathbb{R}^n, \mathbb{V})$ is the space of those $v \in L^{ps}(\mathbb{R}^n, \mathbb{V})$ such that $\mathcal{A}v = 0$ in the sense of distributions. We will further need the following standard non-degeneracy assumption:

$$\text{the symbol of } \mathcal{A}, \text{ seen as a matrix-valued polynomial, has constant rank.} \tag{4.2}$$

Whenever (4.2) holds, we say that $\mathcal{A}$ has *constant rank*. We will not discuss this assumption here but it holds in all of the examples below; the reader may find other characterizations of constant rank operators in [36, 58].

Let $T: X^* \to Y^*$ be a homogeneous sequentially weakly continuous operator. Under the assumption (4.2), such operators were completely characterised in [35], and they are often called *Compensated Compactness quantities*. They can be realised as certain constant-coefficient partial differential operators and so they necessarily satisfy (A3) if one takes the isometries $\sigma_k^{X^*}, \sigma_k^{Y^*}$ to be translations. The following are standard examples of such operators:

(i) $\mathcal{A} = \operatorname{curl}$ and $T = \mathrm{J}$. For this example, take $\mathbb{V} = \mathbb{R}^{n\times n}$ and choose $\mathcal{A}$ in such a way that $\mathcal{A}v = 0$ if and only if $v = \mathrm{D}u$, for some $u: \mathbb{R}^n \to \mathbb{R}^n$. For instance, we may take $(\operatorname{curl} v)_{ijk} = \partial_k v_{ij} - \partial_j v_{ik}$. We also choose $s = n$ and so $X^* = \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$. The only positively n-homogeneous sequentially weakly continuous operator $X^* \to Y^*$ is the Jacobian, and in particular we recover Corollary B.

(ii) $\mathcal{A} = \operatorname{curl}^2$ and $T = \mathrm{H}$. Here $\mathcal{A}$ is chosen similarly to the previous example, but now $\mathcal{A}v = 0$ if and only if $v = \mathrm{D}^2 u$, for some $u: \mathbb{R}^n \to \mathbb{R}$. Again we take $s = n$ and so $X^* = \dot{W}^{2,np}(\mathbb{R}^n, \mathbb{R}^n)$. We may take $T = \mathrm{H}: X^* \to Y^*$ to be the Hessian, and Theorem A shows that it satisfies the open mapping principle.

The two previous examples admit a straightforward generalisation, where one considers s-th order minors (instead of the determinant) and a j-th order curl (instead of $j = 1, 2$).

(iii) $\mathcal{A} = (\mathrm{div}, \mathrm{curl})$ and $T = \langle \cdot, \cdot \rangle$. In this example, $s = 2$ and T is the standard inner product acting on a pair $v \equiv (B, E) \colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$; here, B is thought of as a "magnetic field" and E as an "electric field". As before, Theorem A shows that T satisfies the open mapping principle.

We conclude this subsection by comparing the above example with [21]. There, the authors address the problem of deciding whether Compensated Compactness quantities are surjective, particularly when $p = 1$. Thus Theorem A can be read as saying that openness at zero is a necessary condition for a positive answer to this problem.

4.2 A nonlinear uniform boundedness principle

We also present a nonlinear version of the Uniform Boundedness Principle in the spirit of Theorem A; under certain structural conditions, a family of operators which is pointwise bounded in a ball is uniformly bounded in a sub-ball.

Proposition 4.3. *Let X and Z be Banach spaces and let I be an index set. Suppose the following conditions hold:*

(i) *For every $i \in I$, the mapping $T_i \colon X \to Z$ is such that $u \mapsto \|T_i u\|_Z \colon X \to \mathbb{R}$ is weakly sequentially lower semicontinuous.*

(ii) *There is $\varepsilon > 0$ such that $\sup_{i \in I} \|T_i(u)\|_Z < \infty$ whenever $\|u\|_X \leq \varepsilon$.*

(iii) *For $j \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^X \colon X \to X$ and $\sigma_k^Z \colon Z \to Z$ such that*

$$T_i \circ \sigma_k^X = \sigma_k^Z \circ T_i \qquad \text{for all } i \in I \text{ and } k \in \mathbb{N},$$
$$\sigma_k^X u \rightharpoonup 0 \qquad \text{for all } u \in X.$$

Then there exists $\delta > 0$ such that

$$\sup_{\|u\|_X \leq \delta} \sup_{i \in I} \|T_i u\|_Z < \infty.$$

Proof. By (ii), we may write $\varepsilon \mathbb{B}_X = \cup_{\ell=1}^{\infty} C_\ell$, where $C_\ell \equiv \{u \in \varepsilon \mathbb{B}_X \colon \sup_{i \in I} \|T_i u\|_Z \leq \ell\}$ and (i) shows that each C_ℓ is norm closed. Thus, by the Baire Category Theorem, some C_ℓ contains a closed ball $\bar{B}_\delta(u_0)$.

Let now $\|u\|_X \leq \delta$ and $i \in I$. By (iii), we have $u + \sigma_k^X u_0 = \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u] \in \bar{B}(u_0, \delta)$ and moreover $u + \sigma_k^X u_0 \rightharpoonup u$. So by (i) and again (iii), we have

$$\|T_i u\|_Z \leq \liminf_{k \to \infty} \|T_i \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u]\|_Z = \liminf_{k \to \infty} \|\sigma_k^Z T [u_0 + (\sigma_k^X)^{-1} u]\|_Z \leq \ell.$$

The proof is complete. □

4.3 A trichotomy on atomic decompositions in terms of T

It is conceivable that $\mathrm{J}\colon \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ is not surjective but Theorem 1.3 improves to a *finitary* decomposition of $\mathscr{H}^p(\mathbb{R}^n)$ in terms of Jacobians. In Proposition 4.4, we formulate a rather precise trichotomy about infinitary and finitary decompositions in the setting of Theorem A.

Given T as in Theorem A, if every $f \in Y^*$ can be written as

$$f = \sum_{j=1}^{n} c_j\, T u_j, \qquad c_j \in \mathbb{R},\ u_j \in \mathbb{B}_{X^*}, \tag{4.3}$$

then, following [29], T is said to be $1/n$*-surjective.* If, furthermore,

$$\sum_{j=1}^{n} |c_j| \lesssim \|f\|_{Y^*} \tag{4.4}$$

for all $f \in Y^*$, then T is said to be $1/n$*-open.* Dixon [29] generalised Horowitz's example by constructing, for every $n \in \mathbb{N}$, a continuous $1/n$-surjective bilinear map between Banach spaces which is not $1/n$-open.

In Proposition 4.4 we show that under the assumptions of Theorem A, $1/n$-surjectivity implies $1/n$-openness.

Proposition 4.4. *Suppose X, Y and T satisfy the assumptions of Theorem A. Then one of the following three conditions holds:*

(i) *There exists $n \in \mathbb{N}$ such that formulas (4.3)–(4.4) hold for all $f \in Y^*$ but the set $\{\sum_{j=1}^{n-1} c_j T u_j : c_j \in \mathbb{R},\ u_j \in \mathbb{B}_{X^*}\}$ is meagre in Y^*.*

(ii) $\{\sum_{j=1}^{\infty} c_j\, T u_j : \sum_{j=1}^{\infty} |c_j| < \infty,\ u_j \in \mathbb{B}_{X^*}\} = Y^*$, *but* $\operatorname{span}(T(X^*))$ *is meagre in* Y^*.

(iii) $\{\sum_{j=1}^{\infty} c_j\, T u_j : \sum_{j=1}^{\infty} |c_j| < \infty,\ u_j \in \mathbb{B}_{X^*}\}$ *is meagre in* Y^*.

Proof. We first show that if neither of (ii) and (iii) holds, then (i) does. Since $\operatorname{span}(T(X^*))$ is not meagre in Y^* and can be written as the union of the closed sets

$$D_n \equiv \left\{ \sum_{j=1}^{n} c_j T u_j \colon c_j \in \mathbb{R},\ u_j \in \mathbb{B}_{X^*} \right\},$$

the Baire Category Theorem implies that one of the sets D_n contains a ball. Now (4.3)–(4.4) follow by applying Theorem A to the operator

$$\tilde{T}\colon \mathbb{R}^n \times \mathbb{B}_{X^*}^n \to Y^*, \qquad \tilde{T}\Big(\{c_j\}_{j=1}^n, \{u_j\}_{j=1}^n\Big) \equiv \sum_{j=1}^{n} c_j\, T u_j.$$

Claim (i) follows by choosing the smallest $n \in \mathbb{N}$ such that $T\colon X^* \to Y^*$ is $1/n$-surjective.

Suppose then that (i) and (iii) do not hold; we intend to prove claim (ii). If $\operatorname{span}(T(X^*))$ were non-meagre, the proof above would yield a contradiction. Now, since (iii) was assumed to fail, $\{\sum_{j=1}^{\infty} c_j T u_j \colon \sum_{j=1}^{\infty} |c_j| < \infty,\ u_j \in \mathbb{B}_X\}$ is non-meagre in Y^*. Thus one of the closed sets $\widetilde{D}_\ell \equiv \{\sum_{j=1}^{\infty} c_j T u_j \colon \sum_{j=1}^{\infty} |c_j| \le \ell,\ u_j \in \mathbb{B}_X\}$, $\ell \in \mathbb{N}$, contains a ball. Hence the set $\widetilde{D}_{2\ell} = \widetilde{D}_\ell - \widetilde{D}_\ell$ contains a ball centred at zero and now the claim follows from (A2). The proof of the proposition is complete. □

Remark 4.5. Note that $T\colon L^2(\mathbb{R}^2,\mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R})$, $T(\omega,\eta) \equiv \mathcal{H}\omega\,\mathcal{H}\eta - \eta\,\omega$ satisfies (i), while for all $n \geq 2$ and $p \in [1,\infty)$ the Jacobian from the *inhomogeneous* Sobolev space $W^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ into $\mathscr{H}^p(\mathbb{R}^n)$ satisfies (iii), see [50]. We are not aware of operators satisfying the assumptions of Theorem A and condition (ii) of Proposition 4.4.

5 Applications to the Jacobian equation

This section contains two parts: in the first one, we use Theorem A to prove Theorem C and, in the second one, we prove Theorem D.

5.1 Existence of well-behaved solutions

In this subsection we focus on the case $n = 2$ for simplicity and we assume throughout that $\mathrm{J}\colon \dot{W}^{1,2p}(\mathbb{R}^2,\mathbb{R}^2) \to \mathscr{H}^p(\mathbb{R}^2)$ is surjective. We are particularly interested in the case $p = 1$. Our goal is to illustrate the way in which Theorem A yields the following principle:

the existence of rough solutions implies the existence of well-behaved solutions.

The following is an example a rough solution, and something that we would like to avoid:

Example 5.1 ([52])**.** There is a map $u \in W^{1,2}(\mathbb{R}^2,\mathbb{R}^2)$ such that

$$\mathrm{J}u = 0 \text{ a.e. in } \mathbb{R}^2 \qquad \text{and} \qquad u([0,1]\times\{0\}) = [0,1]^2.$$

In particular, u does not have the Lusin (N) property.

The main result of this subsection is the following theorem, which shows that in some sense it suffices to deal with non-pathological solutions.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and take $f \in \mathscr{H}^1(\mathbb{R}^2)$ such that $f \geq 0$ in Ω. Assume that $\mathrm{J}\colon \dot{W}^{1,2}(\mathbb{R}^2,\mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2)$ is onto. Then there is a solution $u \in \dot{W}^{1,2}(\mathbb{R}^2,\mathbb{R}^2)$ of* (1.2) *such that:*

(i) *u is continuous in Ω;*

(ii) *u has the Lusin (N) property in Ω.*

(iii) *$\int_{\mathbb{R}^2} |\mathrm{D}u|^2 \,\mathrm{d}x \leq C\|f\|_{\mathscr{H}^1}$ with $C > 0$ independent of f.*

In particular, u satisfies the change of variables formula (2.3)*. Moreover, let $\Omega' \subseteq \Omega$ be an open set such that $f = 0$ a.e. in Ω'. Then:*

(iv) *for any set $E \subset \Omega'$, we have $u(\partial E) = u(\overline{E})$;*

(v) *for $y \in u(\Omega')$, if C denotes a connected component of $u^{-1}(y) \cap \Omega'$ then C intersects $\partial\Omega'$.*

Before proceeding with the proof, we note that (iv) is a type of degenerate monotonicity which had already appeared in the study of the hyperbolic Monge–Ampère equation [18, 47].

Proof. The point of the proof is to perturb f appropriately; then the solution u is obtained as a limit of mappings of integrable distortion.

Let B^+ be a ball containing Ω and let B^- be another ball, disjoint from Ω, and with the same volume as B^+. Consider the perturbations

$$f_\varepsilon \equiv f + \varepsilon a, \qquad a \equiv \chi_{B^+} - \chi_{B^-},$$

which satisfy $f_\varepsilon > 0$ a.e. in Ω. Clearly $a \in \mathscr{H}^1(\mathbb{R}^2)$, being bounded, compactly supported and with zero mean. Hence $f_\varepsilon \to f$ in $\mathscr{H}^1(\mathbb{R}^2)$ and, from Corollary B, we see that we can choose solutions u_ε of $\mathrm{J}u_\varepsilon = f_\varepsilon$ such that $\int_{\mathbb{R}^2} |\mathrm{D}u_\varepsilon|^2 \leq C\|f_\varepsilon\|_{\mathscr{H}^1}$ for all $\varepsilon > 0$. Since the maps u_ε have finite distortion, we can apply Theorem 2.10(i) to conclude that the family (u_ε) is equicontinuous. Hence, upon normalising the maps so that $u_\varepsilon(x_0) = 0$ for some fixed $x_0 \in \Omega'$, and up to taking subsequences, (u_ε) converges both locally uniformly in Ω and weakly in $\dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ to a limit u. This already proves (i) and (iii).

To prove (ii), we note that each u_ε satisfies (2.2), c.f. Proposition 2.11. Since u is the uniform limit of the sequence (u_ε), u also satisfies (2.2) and (ii) follows from Proposition 2.7.

For (iv), note that $\varepsilon \leq f_\varepsilon$ in Ω and so each map u_ε, having integrable distortion, is open; it follows that $\partial u_\varepsilon(E) \subseteq u_\varepsilon(\partial E)$. Suppose, for the sake of contradiction, that there is $y \in u(\overline{E}) \backslash u(\partial E)$. On the one hand, there is some $\delta > 0$ such that, for all ε small enough,

$$B_\delta(y) \cap \partial u_\varepsilon(\operatorname{int} E) \subset B_\delta(y) \cap u_\varepsilon(\partial E) = \emptyset;$$

on the other hand, since $y \in u(\operatorname{int} E)$, for all ε small enough,

$$B_\delta(y) \cap u_\varepsilon(\operatorname{int} E) \neq \emptyset.$$

It follows that $B_\delta(y) \subseteq u_\varepsilon(\operatorname{int} E)$. We also have that $|u_\varepsilon(\operatorname{int} E)| \to 0$ as $\varepsilon \to 0$: by the change of variables formula,

$$|u_\varepsilon(\operatorname{int} E)| \leq \int_{u_\varepsilon(\operatorname{int} E)} \mathcal{N}(y, u_\varepsilon, \operatorname{int} E)\,\mathrm{d}y = \int_E \mathrm{J}u_\varepsilon = \varepsilon|E| \to 0.$$

Thus, since $|B_\delta(y)| \leq |u_\varepsilon(E)|$, a contradiction is reached by sending $\varepsilon \to 0$.

Finally, (v) follows from (iv), as shown for instance in [47, Lemma 2.10]. □

In view of the change of variables formula, it is useful to control the multiplicity function. For the following proposition we again assume that the Jacobian is surjective.

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let $Y \equiv \{f \in \mathscr{H}^p(\mathbb{R}^2) : f \geq c \text{ a.e. in } \Omega\}$, where $c > 0$. Suppose that $f_j \in Y$ is a sequence converging weakly to f in $\mathscr{H}^p(\mathbb{R}^2)$. For any maps $u_j \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $\mathrm{J}u_j = f_j$ and the a priori estimate* (1.4)*, we have that*

$$\sup_j \sup_{y \in u_j(\Omega')} \mathcal{N}(y, u_j, \Omega') < \infty, \qquad \text{whenever } \Omega' \Subset \Omega.$$

Proof. We claim that the sequence u_j is equicontinuous and converges to $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$, a solution of $\mathrm{J}u = f$, uniformly in Ω'. Once the claim is proved, the conclusion follows: u has integrable distortion in Ω and so by Theorem 2.10(v) it is at most m-to-one in Ω', for some $m \in \mathbb{N}$. Thus, for all j sufficiently large, u_j is also at most m-to-one in Ω': if not, there are arbitrarily large j and points $x_1^{(j)},\dots,x_{m+1}^{(j)}\in\Omega'$ such that $u_j(x_i^{(j)})=y$ for some $y\in\mathbb{R}^n$ and all $i\in\{1,\dots,m+1\}$. By compactness, we can further assume that $x_i^{(j)}\to x_i$ for $i=1,\dots,m+1$. However, there are at least two different points $y_1\neq y_2$ such that

$$\{y_1,y_2\}\subset u(\{x_1,\dots,x_{m+1}\});$$

for the sake of definiteness, say $u(x_1)=y_1, u(x_2)=y_2$. Let $\varepsilon<|y_1-y_2|$ and take j sufficiently large so that, for $i=1,2$,

$$|u_j(x_i^{(j)})-u_j(x_i)|<\frac{\varepsilon}{4},\qquad |u_j(x_i)-u(x_i)|<\frac{\varepsilon}{4};$$

this is possible from equicontinuity of the sequence u_j and the fact that it converges to u uniformly. The triangle inequality gives $|y_1-y_2|=|u(x_1)-u(x_2)|<\varepsilon$, a contradiction.

To prove the claim, we assume that the Jacobian is surjective and we use Corollary B. If $p>1$ we appeal to Morrey's inequality,

$$[u_j]_{C^{0,1-2/p}(\mathbb{R}^2)}\lesssim_p\|\mathrm{D}u_j\|_{L^{2p}(\mathbb{R}^2)}\leq C,$$

while for $p=1$ we use Theorem 2.10(i) instead. Either way, after normalizing the maps so that $u_j(x_0)=0$, where $x_0\in\Omega$, we see that the sequence (u_j) is precompact in the local uniform topology over Ω'. Hence we may assume that u_j converges to some map $u\in\dot W^{1,2p}(\mathbb{R}^2,\mathbb{R}^2)$ uniformly in Ω' and also weakly in $\dot W^{1,2p}(\mathbb{R}^2,\mathbb{R}^2)$. □

5.2 Energy minimisers with prescribed Jacobian

For each $f\in\mathscr{H}^p(\mathbb{R}^n)$, we define the *$p$-energy of f* as

$$\mathcal{E}_p(f)\equiv\inf\left\{\int_{\mathbb{R}^n}|\mathrm{D}v|^{np}\,\mathrm{d}x : v\in\dot W^{1,np}(\mathbb{R}^n,\mathbb{R}^n)\text{ satisfies }\mathrm{J}v=f\text{ a.e. in }\mathbb{R}^n\right\}.$$

Note that $f\in\mathscr{H}^p(\mathbb{R}^n)$ has a solution in $\dot W^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ if and only if $\mathcal{E}_p(f)<\infty$; in particular,

$$\mathcal{E}_p\colon\mathscr{H}^p(\mathbb{R}^n)\to[0,+\infty)\tag{5.1}$$

if and only if $\mathrm{J}\colon\dot W^{1,np}(\mathbb{R}^n,\mathbb{R}^n)\to\mathscr{H}^p(\mathbb{R}^n)$ is onto. We use the following terminology [43]:

Definition 5.4. Given $f\in\mathscr{H}^p(\mathbb{R}^n)$, we say that $u\in\dot W^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ is a *p-energy minimiser* for f if

$$\int_{\mathbb{R}^n}|\mathrm{D}u|^{np}\,\mathrm{d}x=\mathcal{E}_p(f)\qquad\text{and}\qquad \mathrm{J}u=f\text{ a.e. in }\mathbb{R}^n.$$

If all p-energy minimisers v of f can be written as $v=Q_vu$, for some $Q_v\in\mathrm{SO}(n)$, then we say that the p-energy minimisers for f are *unique modulo rotations.*

Recall that a map in $\dot W^{1,np}$ is defined only modulo constants. For simplicity, we will typically assume that p-energy minimisers are normalised so that they map zero to zero.

We begin with a simple abstract lemma:

Lemma 5.5. *Assume* (5.1) *holds. Then $\mathcal{E}_p\colon\mathscr{H}^p(\mathbb{R}^n)\to[0,+\infty)$ is sequentially weakly lower semicontinuous. In particular, for any bounded set $B\subset\mathscr{H}^p(\mathbb{R}^n)$, the set of weak continuity points of $\mathcal{E}_p\colon B\to[0,+\infty)$ is residual in the weak topology of B.*

Proof. The sequential weak lower semicontinuity follows immediately from the sequential weak continuity of the Jacobian and the convexity of the norm in $\dot{W}^{1,2p}$; then $\mathcal{E}_p|_B$ is also sequentially weak lower semicontinuous. Since the weak topology is metrisable on bounded sets, Baire's theorem (see for instance [46, Exercise (24.16)]) then asserts that $\mathcal{E}_p$ is a Baire-1 function and hence its set of weak continuity points is residual in B. □

Remark 5.6. When $B \subset \mathscr{H}^p(\mathbb{R}^n)$ is a closed ball, and if $\mathcal{E}_p$ is finite-valued, we must have that all weak continuity points of $\mathcal{E}_p$ are located in ∂B. Indeed, if there is a weak continuity point $f \in B \backslash \partial B$ of $\mathcal{E}_p$, then we derive a contradiction as follows: take a sequence $f_j \in B$ which converges to f weakly but not strongly. By weak continuity of $\mathcal{E}_p$ at f, and passing to a subsequence if need be, we can find a sequence of p-energy minimisers u_j of f_j such that $\mathrm{D}u_j \rightharpoonup \mathrm{D}u$ in $L^{np}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and $\|\mathrm{D}u_j\|_{L^{np}} \to \|\mathrm{D}u\|_{L^{np}}$. Since Lebesgue spaces are uniformly convex they have the Radon–Riesz property, that is, $\mathrm{D}u_j \to \mathrm{D}u$ strongly in L^{np}, see [14, Proposition 3.32] and [25]. Hence we deduce that $f_j \to f$ strongly in $L^p(\mathbb{R}^n)$, which is absurd.

Our interest in the continuity points of $\mathcal{E}_p$ comes from the following straightforward lemma:

Lemma 5.7. *Let u_j be a p-energy minimiser for f_j. Suppose that f is a strong (respectively weak) continuity point of $\mathcal{E}_p$ and that $f_j \to f$ (respectively $f_j \rightharpoonup f$) in $L^p(\mathbb{R}^n)$. Then, up to a subsequence, $u_j \to u$ in $\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ and u is a p-energy minimiser for f.*

We would like to use methods similar to the ones in Section 5.1 to deduce that p-energy minimisers are well-behaved solutions. One of the main difficulties with this approach, however, is that we do not know how to find strong continuity points of $\mathcal{E}_p$ and, in light of Remark 5.6, the situation with weak continuity points is not very satisfying: if $f_j \rightharpoonup f$ in L^p and, for all large j, $\|f_j\| > \|f\|$, then it follows that $\mathcal{E}_p(f_j) \not\to \mathcal{E}_p(f)$. Hence one is forced to consider perturbations with norm not exceeding that of f.

It may be that the difficulty in finding strong continuity points of $\mathcal{E}_p$ is genuine; the main result of this subsection certainly shows that the multi-valued map

$$f \mapsto \left\{u : \|\mathrm{D}u\|_{L^{np}}^{np} = \mathcal{E}_p(f)\right\}$$

is rather badly behaved. Indeed, we will now prove Theorem D, which we restate here:

Theorem 5.8. *Fix $1 \leq p < \infty$. There is a radially symmetric function $f \in \mathscr{H}^p(\mathbb{R}^n)$ which has uncountably many p-energy minimisers, modulo rotations.*

A more informative statement can be found in Corollary 5.13, at the end of the section. The proof of Theorem 5.8 relies mostly on elementary tools and the most sophisticated result that we use is the following:

Theorem 5.9 (Sierpiński)**.** *Let (X_n) be disjoint closed sets such that $I = \bigcup_{n\in\mathbb{N}} X_n$, where $I = [a, b] \subset \mathbb{R}$. There is at most one $n \in \mathbb{N}$ such that X_n is non-empty.*

Theorem 5.9 is only needed to obtain uncountably many distinct minimisers, as non-uniqueness follows already from more elementary means. We also note that Theorem 5.9 holds more generally for a compact, connected Hausdorff space, see e.g. [31, Theorem 6.1.27]. In the case of an interval the proof is simple and so we give it here for the sake of completeness:

Proof. Take $Y \equiv \bigcup_n \partial X_n = I \backslash \bigcup_n \operatorname{int}(X_n)$, which is closed, thus a complete metric space.

We observe that the set Y has empty interior in I, i.e. any open interval L contains an open set U disjoint from Y. Indeed, from the Baire Category Theorem we see that there is an open set $U \subseteq L$ and some X_m which is dense in U. Since X_m is closed, we must have $U \subseteq \operatorname{int} X_m$ and thus U is disjoint from Y.

By the Baire Category Theorem there is also some open subinterval J of I and some $n \in \mathbb{N}$ such that ∂X_n is dense in $Y \cap J$. Since ∂X_n is closed we have $\partial X_n \cap J = Y \cap J$. Thus $(Y \backslash \partial X_n) \cap J = \emptyset$.

Suppose now that $X_n \neq I$. It follows that J intersects $Y \backslash \partial X_n$. Indeed, since Y has empty interior in I, J intersects $I \backslash X_n$ and so it intersects $\operatorname{int}(X_k)$ for some k. Actually, J must intersect ∂X_k: otherwise, $\operatorname{int}(X_k) \cap J$ is non-empty, open and closed in J, thus $\operatorname{int} X_k = J$, since J is connected; clearly this is impossible, since X_k is disjoint from X_n. So we proved that J intersects $Y \backslash \partial X_n$, contradicting the previous paragraph. □

We are now ready to begin the proof of Theorem 5.8, whose core idea is contained in the following lemma.

Lemma 5.10. *Let u be a p-energy minimiser for a radially symmetric function $f \in \mathscr{H}^p(\mathbb{R}^2)$. For $\alpha_0 \in [0, 2\pi]$, consider the set*

$$X_{\alpha_0} \equiv \{\alpha \in [0, 2\pi] : u_\alpha = u_{\alpha_0} \text{ modulo rotations}\}, \qquad \text{where } u_\alpha(z) \equiv u(e^{i\alpha} z). \tag{5.2}$$

Assume that $f \in C^0(B_R)$ has a sign. If $X_{\alpha_0} = [0, 2\pi]$ then there is $k \in \mathbb{Z}$ such that

$$u(z) = \phi_k(z) \quad \text{in } B_R, \text{ modulo rotations.}$$

Proof. If $X_{\alpha_0} = [0, 2\pi]$ then, for any $\alpha \in [0, 2\pi]$ and $z \in B_R$, we have $|u(e^{i\alpha} z)| = |u(z)|$; that is, circles in B_R, centred at zero, are mapped to circles centred at zero.

For each $r \in (0, R)$, we have $0 \notin u(S_r)$. Indeed, for each ball $B \Subset B_R$, there is $c = c(B) > 0$ such that $f \geq c$ in B (or $f \leq -c$, but by reversing orientations we can always consider the first case without loss of generality). Thus, in B_r, u is a map of integrable distortion and so, by Theorem 2.10, it is both continuous and open. Therefore $\partial(u(B_r)) \subseteq u(\partial B_r) = u(S_r)$ and we see that $u(S_r) \neq \{0\}$. Since $u(S_r)$ is a circle, we conclude that $0 \notin u(S_r)$.

By Proposition 2.1 we may write

$$u(r, \theta) = \psi(r, \theta) e^{i\gamma(r, \theta)} \tag{5.3}$$

where $\psi \in W^{1,2p}(B_R, [0, \infty))$, $\gamma \in W^{1,2p}_{\text{loc}}(B_R, \mathbb{R})$. Moreover, ψ is continuous in B_R, γ is continuous in $B_R \setminus \{0\}$ and further $\gamma(r, 2\pi) - \gamma(r, 0) \in 2\pi\mathbb{Z}$ for a.e. $r \in (0, R)$. From the representation (5.3), it is not difficult to formally derive the formulae

$$\mathrm{J}u = \frac{1}{2r} \frac{\partial(\psi^2, \gamma)}{\partial(r, \theta)} = \frac{1}{2r} \Big(\partial_r(\psi^2) \partial_\theta \gamma - \partial_\theta(\psi^2) \partial_r \gamma \Big), \tag{5.4}$$

$$|\mathrm{D}u|^2 = |\partial_r \psi|^2 + |\psi \partial_r \gamma|^2 + \frac{|\partial_\theta \psi|^2}{r^2} + \frac{|\psi \partial_\theta \gamma|^2}{r^2}. \tag{5.5}$$

Due to the regularity of ψ and γ, the right-hand sides in (5.4)–(5.5) define locally integrable functions and so a density argument shows that these formulae hold a.e. in $B_R(0)$.

For $r < R$, $u(S_r) = S_{r'}$, that is, $\psi(r,\theta)$ is independent of θ. Thus (5.4) reduces to

$$\partial_r(\psi^2)\partial_\theta\gamma = 2rf(r), \tag{5.6}$$

which is valid for almost every $(r,\theta) \in B_r \setminus \{0\}$. Since both ψ and the right-hand side are independent of θ we must have $\gamma(r,\theta) = 2\pi k\theta + \beta(r)$ and additionally there is the compatibility constraint $k \in \mathbb{Z}$. Since u is a p-energy minimiser, (5.5) readily implies that β is constant. We integrate both sides of (5.6), using $\psi(0) = 0$, to find

$$\psi(r)^2 = \frac{1}{k}\int_0^r 2sf(s)\,\mathrm{d}s \qquad \text{for } r < R.$$

Thus, modulo rotations, $u = \phi_k$ in B_R. □

In fact, the same argument applied in an annulus $\mathbb{A}(R_0, R)$ gives the following variant:

Lemma 5.11. *Consider the setup of Lemma 5.12, but replace B_R by $\mathbb{A}(R_0, R)$. Then there is $k \in \mathbb{Z}$ and $c \in \mathbb{R}$ such that, in $\mathbb{A}(R_0, R)$,*

$$u(z) = \psi(r)e^{2\pi ik\theta} \text{ modulo rotations}, \qquad \text{where } \psi(r)^2 = \int_{R_0}^r 2sf(s)\,\mathrm{d}s + c.$$

We now combine the previous two lemmas.

Lemma 5.12. *There is a radially symmetric $f \in \mathscr{H}^p(\mathbb{R}^2)$, admitting a p-energy minimiser u, for which we have $X_0 \neq [0, 2\pi]$, where X_0 is as in* (5.2)*.*

Proof. We take a function $f \colon \mathbb{R}^2 \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} &f \in C^1(\mathbb{R}^n) \text{ is radially symmetric,} \\ &\int_{B_2} f\,\mathrm{d}x = \int_{\mathbb{R}^2} f\,\mathrm{d}x = 0 \\ &f(r) < 0 \text{ if } 0 < r < 1, \quad f > 0 \text{ if } 1 < r < 2, \quad f(r) = (4-r)^+ \text{ if } 3 < r. \end{aligned} \tag{5.7}$$

By [48, Theorem 4], there is $v \in C^1(\overline{B_4}, \mathbb{R}^2)$ such that $\mathrm{J}v = f$ and $v = 0$ on ∂B_4; in particular, by extending v by zero outside B_4, we have $v \in W^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$. Since the $2p$-Dirichlet energy is convex, the Direct Method, combined with the weak continuity of the Jacobian, shows that f has at least one p-energy minimiser and we call it u, using it to define the sets in (5.2).

Suppose, for the sake of contradiction, that $X_0 = [0, 2\pi]$. Using Lemmas 5.10 and 5.11, we deduce that there are angles $\alpha, \alpha' \in [0, 2\pi)$, numbers $k, k' \in \mathbb{Z}$ and $c \in \mathbb{R}$ such that

$$u = e^{i\alpha}\phi_k \text{ in } B_1, \qquad u = e^{i\alpha'}\left(\psi(r)e^{2\pi ik'\theta}\right) \text{ in } \mathbb{A}(1,2),$$

where, for $r \in (1,2)$,

$$\psi(r)^2 = \frac{1}{k'}\int_1^r 2sf(s)\,\mathrm{d}s + c.$$

In the notation of Definition 2.3, we must have

$$e^{i\alpha + 2\pi ik\theta}\frac{\rho(1)}{\sqrt{|k|}} \equiv \mathrm{Tr}_{S_1} u_{|B_1} = \mathrm{Tr}_{S_1} u_{|\mathbb{A}(1,2)} \equiv e^{i\alpha' + 2\pi ik'\theta}c$$

in $L^{2p}(S_1)$. It is easy to conclude that $\alpha = \alpha'$, $k = k'$ and $c = \rho(1)/\sqrt{|k|}$, and so, modulo rotations, actually $u = \phi_k$ in B_2. It is now easy to verify directly that, for f as in (5.7), we have

$$\int_0^2 |\dot{\rho}(r)|^2 \, \mathrm{d}r = +\infty,$$

and so by Lemma 2.4 $u \notin W^{1,2}(B_2, \mathbb{R}^2)$, which is a contradiction. Alternatively, one can infer that $u \notin W^{1,2}(B_2, \mathbb{R}^2)$ from [50, Theorem 3.4]. □

Proof of Theorem 5.8. Let f and u be as in Lemma 5.12. For each $\alpha \in [0, 2\pi]$, it is easy to check that the set X_α is closed. We may write, for some index set A,

$$[0, 2\pi] = \bigcup_{\alpha \in A} X_\alpha, \qquad \text{where the union is disjoint.}$$

For distinct $\alpha, \alpha' \in A$, X_α and $X_{\alpha'}$ correspond to distinct equivalence classes of p-energy minimisers, and so by Lemma 5.12 we must have $\# A > 1$. But now Theorem 5.9 shows that A must be uncountable. □

We also note that the proof of Lemma 5.10 yields the following corollary:

Corollary 5.13. *Let $f \in \mathscr{H}^p(\mathbb{R}^2)$ be radially symmetric and suppose u is its unique p-energy minimiser, modulo rotations. If u is continuous then $u = \phi_k$ for some $k \in \mathbb{Z}$.*

Clearly the continuity assumption is not restrictive if $p > 1$.

Proof. As in the proof of Lemma 5.10 we conclude that u maps circles centred at zero to circles and that $(r, \theta) \mapsto |u(re^{i\theta})|$ is independent of θ. Thus we write simply $|u(r)|$.

We show that the set $\{r \in (0, \infty) \colon |u(r)| > 0\}$ is connected. Suppose, by way of contradiction, that there are $r_1 < r_2 < r_3$ such that $|u(r_1)|, |u(r_3)| > 0$ but $|u(r_2)| = 0$. We get another p-energy minimiser for f by setting

$$v(z) = \begin{cases} u(z), & |z| \leq r_2, \\ e^{i\pi} u(z), & |z| > r_2, \end{cases}$$

contradicting the assumption that the p-energy minimiser for f is unique modulo rotations.

Thus we can write, for some $0 \leq R_1 \leq R_2 \leq \infty$,

$$\big\{r \in (0, \infty) \colon |u(r)| > 0\big\} = (R_1, R_2)$$

and clearly we must have $f(r) = 0$ if $r \notin (R_1, R_2)$. Thus $\phi_k(z) = 0$ if $r \notin (R_1, R_2)$ and so $u = \phi_k$ outside $\mathbb{A}(R_1, R_2)$. Moreover, can use Lemma 5.11 to conclude that $u = \phi_k$ in $\mathbb{A}(R_1, R_2)$, modulo rotations, and the conclusion follows. □

6 A general nonlinear open mapping principle for scale-invariant problems

The main result of this section is Theorem 6.2, which is a generalisation of Theorem A to a wider class of translation-invariant, scaling-invariant PDEs. Section 6.2 illustrates the way in which Theorem 6.2 can be applied to some physical nonlinear equations.

6.1 A more general nonlinear open mapping principle

We begin by formulating a model problem abstractly as follows:

$$\text{if } g-1 \in Y^*, \text{ does there exist } v \text{ with } \mathrm{J}v = f \text{ and } v - \mathrm{id} \in X^*? \tag{6.1}$$

Here X^* and Y^* are suitably chosen function spaces. In smooth domains $\Omega \subsetneq \mathbb{R}^n$, examples of (6.1) include the Dirichlet problem for the Jacobian equation, that is,

$$\begin{cases} \mathrm{J}v = g & \text{in } \Omega, \\ v = \mathrm{id} & \text{on } \partial\Omega; \end{cases} \tag{6.2}$$

the condition $g - 1 \in Y^*$ is codified in the compatibility condition

$$\int_\Omega (g-1)\,\mathrm{d}x = 0.$$

We return to the abstract formulation (6.1). When $n = 2$, denoting $f \equiv g - 1$ and $u \equiv v - \mathrm{id}$ we get the following question, equivalent to (6.1):

$$\text{if } f \in Y^*, \text{ does there exist } u \in X^* \text{ with } Tu \equiv \mathrm{J}u + \operatorname{div} u = f? \tag{6.3}$$

The latter formulation has the advantage that X^* and Y^* are vector spaces, which makes the problem more amenable to scaling arguments.

In Example 6.1 we discuss a representative special case of (6.3). Here T does not map X^* into Y^* and we therefore need to choose a set $D \subsetneq X^*$ as the domain of definition of T.

Example 6.1. Let $X^* = \dot W^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$ and $Y^* = L^p(\mathbb{R}^2)$ with $p \in [2,\infty)$ and $q \in [p, 2p]$. Since $T = \mathrm{J} + \operatorname{div}$ does not map $\dot W^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$, it is natural to set

$$D = \{u \in \dot W^{1,q}(\mathbb{R}^2,\mathbb{R}^2) \colon Tu \in L^p(\mathbb{R}^2)\}$$

and study the range of

$$T = \mathrm{J} + \operatorname{div} \colon D \to Y^*. \tag{6.4}$$

Note that we may write $T \circ \tau_\lambda^D = \tau_\lambda^{Y^*} \circ T$ for all $\lambda > 0$, where

$$\tau_\lambda^D u(x) = u_\lambda(x) \equiv \lambda u\left(\frac{x}{\lambda}\right), \qquad \tau_\lambda^{Y^*} f(x) = f_\lambda(x) \equiv f\left(\frac{x}{\lambda}\right)$$

give multiples of isometries:

$$\|\tau_\lambda^D u\|_{\dot W^{1,q}} = \lambda^{2/q}\|u\|_{\dot W^{1,q}}, \qquad \|\tau_\lambda^{Y^*} f\|_{L^p} = \lambda^{2/p}\|f\|_{L^p}$$

for all $u \in D$, $f \in Y^*$ and $\lambda > 0$.

Since the set D contains the proper dense subspace $C_c^\infty(\mathbb{R}^2,\mathbb{R}^2)$, it is neither weakly nor strongly closed in $\dot W^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$. This difficulty is reflected in the somewhat awkward assumption $(\widehat{\mathrm{A4}})$ of Theorem 6.2 below.

Before formulating the result recall that when a direct sum of Banach spaces $X = \oplus_{i=1}^m X_i$ is endowed with the norm $\|w\|_X \equiv \sum_{i=1}^M \|w_i\|_{X_i}$, the dual norm of $X^* = \oplus_{i=1}^m X_i^*$ is of the form $\|u\|_{X^*} = \max_{1\le i\le m} \|u_i\|_{X_i^*}$.

Theorem 6.2. *Let $X_1, \dots, X_I$ and $Y_1, \dots, Y_J$ be Banach spaces and denote $X = \oplus_{i=1}^I X_i$ and $Y = \oplus_{j=1}^J Y_j$. Suppose $\mathbb{B}_{X^*}$ is sequentially weak* compact and $0 \in D \subset X^*$.*

We make the following assumptions:

($\widehat{\text{A1}}$) *$T\colon D \to Y^*$ is a weak*-to-weak* sequentially continuous operator.*

($\widehat{\text{A2}}$) *For $\lambda > 0$, there exist bijections $\tau_\lambda^D\colon D \to D$ and $\tau_\lambda^{Y^*}\colon Y^* \to Y^*$ such that*

$$\begin{aligned} &T \circ \tau_\lambda^D = \tau_\lambda^{Y^*} \circ T && \text{for all } \lambda > 0,\\ &\|(\tau_\lambda^D u)_i\|_{X_i^*} = \lambda^{r_i}\|u_i\|_{X_i^*} && \text{for all } \lambda > 0,\ i = 1, \dots, I,\ u \in X^*,\\ &\|(\tau_\lambda^{Y^*} f)_j\|_{Y_j^*} = \lambda^{s_j}\|f_j\|_{Y_j^*} && \text{for all } \lambda > 0,\ j = 1, \dots, J,\ f \in X^*, \end{aligned}$$

where $0 < r_1 \le \dots \le r_I$ and $0 < s_1 \le \dots \le s_J$.

($\widehat{\text{A3}}$) *There exist sequences of isometric bijections $\sigma_k^D\colon D \to D$ with $\sigma_k^D(0) = 0$ and isometric isomorphisms $\sigma_k^{Y^*}\colon Y^* \to Y^*$ such that*

$$T \circ \sigma_k^D = \sigma_k^{Y^*} \circ T \qquad \text{for all } k \in \mathbb{N}, \qquad\qquad \sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \text{for all } f \in Y^*.$$

($\widehat{\text{A4}}$) *For $\ell \in \mathbb{N}$, the sets $D_\ell \equiv \{u \in D\colon \|u\|_{X^*} \le \ell,\ \|Tu\|_{Y^*} \le \ell\}$ are weakly* sequentially closed in X^*.*

The following conditions are then equivalent:

(i) *$T(D)$ is non-meagre in Y^*.*

(ii) *$T(D) = Y^*$.*

(iii) *T is open at the origin.*

(iv) *For every $f \in Y^*$ there exists $u \in D$ such that*

$$Tu = f, \qquad \begin{cases} \sum_{i=1}^I \|u_i\|_{X_i^*}^{s_J/r_i} \le C\|f\|_{Y^*}, & \|f\|_{Y^*} \le 1,\\ \sum_{i=1}^I \|u_i\|_{X_i^*}^{s_1/r_i} \le C\|f\|_{Y^*}, & \|f\|_{Y^*} > 1. \end{cases} \tag{6.5}$$

Proof. We first show (i) $\Rightarrow$ (iii), so assume (i) holds. Write $D = \cup_{\ell=1}^\infty D_\ell$ and note that $T(D) = \cup_{\ell=1}^\infty T(D_\ell)$. Since $\mathbb{B}_{X^*}$ is weak* sequentially compact and $T\colon D \to Y^*$ is weak*-to-weak* sequentially continuous, the sets $T(D_\ell)$ are closed in Y and, therefore, complete. By the Baire Category Theorem, one of the sets $T(D_\ell)$ contains a ball $\bar{B}_\eta(f_0)$. Clearly $\eta \le \ell$. We first show that

$$T(D \cap \ell\mathbb{B}_{X^*}) \supset \eta\mathbb{B}_{Y^*}. \tag{6.6}$$

Suppose $f \in Y^*$ with $\|f\|_{Y^*} \le \eta$. Since the maps $\sigma_k^{Y^*}\colon Y^* \to Y^*$ are isometries, we get $f_0 + (\sigma_k^{Y^*})^{-1} f \in \bar{B}_\eta(f_0) \subset T(D_\ell)$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, choose $u_k \in D_\ell$ such that $Tu_k = f_0 + (\sigma_k^{Y^*})^{-1} f$. By ($\widehat{\text{A3}}$), each σ_k^D maps D_ℓ into D_ℓ. Thus, passing to a subsequence as in the proof of Theorem A, and using ($\widehat{\text{A4}}$), $\sigma_k^D u_k \rightharpoonup u \in D_\ell$; by ($\widehat{\text{A1}}$) we get $Tu = f$. Thus (6.6) is proved.

We are ready to show openness of T at zero. Let $\varepsilon > 0$; our aim is to find $\delta > 0$ such that $T(D \cap \varepsilon\mathbb{B}_{X^*}) \supset \delta\mathbb{B}_{X^*}$. We first note that for each $\lambda > 0$ we have

$$\tau_\lambda^D(D \cap \ell\mathbb{B}_{X^*}) = \{u \in D\colon \|u_i\|_{X_i^*} \le \lambda^{r_i}\ell \text{ for } i = 1, \dots, I\}.$$

By choosing $\lambda = \min_{1\le i\le I}(\varepsilon/\ell)^{1/r_i}$ we get $\max_{1\le i\le I}\lambda^{r_i}\ell \le \varepsilon$ so that

$$T(D\cap \varepsilon\mathbb{B}_{X^*}) \supset T(\tau_\lambda^D(D\cap \ell\mathbb{B}_{X^*})) = \tau_\lambda^{Y^*}T(D\cap \ell\mathbb{B}_{X^*}).$$

By using (6.6) and selecting $\delta = \min_{1\le i\le I}\min_{1\le j\le J}\eta(\varepsilon/\ell)^{s_j/r_i}$ we get

$$\tau_\lambda^{Y^*}T(D\cap \ell\mathbb{B}_{X^*}) \supset \tau_\lambda^{Y^*}(\eta\mathbb{B}_{Y^*}) = \lambda^{s_1}\eta\mathbb{B}_{Y_1^*}\times\cdots\times\lambda^{s_J}\eta\mathbb{B}_{Y_J^*} \supset \delta\mathbb{B}_{Y^*},$$

as wished.

We now prove (iii) ⇒ (iv), so as above take some $\varepsilon > 0$ and get $\delta > 0$ in such a way that $\delta\mathbb{B}_{Y^*} \subset T(D\cap\varepsilon\mathbb{B}_{X^*})$. Assume, without loss of generality, that $\delta \le 1$. Let $f \in Y^*$ and define $\lambda > 0$ via

$$\|f\|_{Y^*} \equiv \mu = \min_{1\le j\le J}\lambda^{s_j}\delta = \begin{cases}\lambda^{s_J}\delta, & \mu\le\delta,\\ \lambda^{s_1}\delta, & \mu>\delta.\end{cases}$$

In either case, denote $\mu = \lambda^{s_{j_0}}\delta$. Then

$$\begin{aligned} f \in \tau_\lambda^{Y^*}(\delta\mathbb{B}_{Y^*}) \subset \tau_\lambda^{Y^*}T(D\cap\varepsilon\mathbb{B}_{X^*}) &= T\tau_\lambda^D(D\cap\varepsilon\mathbb{B}_{X^*}) \\ &= T\{u\in D\colon \|u_i\|_{X_i^*}\le\lambda^{r_i}\varepsilon \text{ for } i=1,\dots,I\}. \end{aligned}$$

Suppose now $u \in D$ satisfies $\|u_i\|_{X_i^*} \le \lambda^{r_i}\varepsilon$ for $i = 1,\dots,I$. Then, for all i,

$$\|u_i\|_{X_i^*}^{s_{j_0}/r_i} \le \lambda^{s_{j_0}}\varepsilon^{s_{j_0}/r_i} \le \frac{\varepsilon^{s_{j_0}/r_i}}{\delta}\mu.$$

We conclude that

$$f \in T\left\{u\in D\colon \|u_i\|_{X_i^*}\le\lambda^{r_i}\varepsilon \text{ for all } i\right\} \subset T\left\{u\in D\colon \sum_{i=1}^I\|u_i\|_{X_i^*}^{s_{j_0}/r_i}\le C\mu\right\},$$

where

$$C = \sum_{i=1}^I\frac{\varepsilon^{s_{j_0}/r_i}}{\delta},$$

which yields (6.5) in the cases $\|f\|_{Y^*}\le\delta$ and $\|f\|_{Y^*} > 1$. In the bounded regime $\delta < \|f\|_{Y^*} \le 1$, one obviously has $\lambda^{s_1}\approx_\delta\lambda^{s_J}$ so that (6.5) holds for all f.

We conclude the proof of the theorem by noting that (iv) ⇒ (ii) ⇒ (i). □

Remark 6.3. Inspection of the proof of Theorem 6.2 shows that, in the statement of the theorem, one may replace all occurrences of Y^* with K, where $K \subset Y^*$ is a closed convex cone. Recall that K is said to be a *cone* if $af \in K$ whenever $a > 0$ and $f \in K$.

Such a generalisation is occasionally useful, since it may be interesting to consider smaller data sets. For instance, in the case Question 1.2, natural examples include the set of radially symmetric data $K = \{f \in \mathscr{H}^p(\mathbb{R}^n) : f(x) \equiv f(|x|)\}$ and, when $p > 1$, the set of non-negative data $K = \{f \in L^p(\mathbb{R}^n) : f \ge 0\}$.

Returning to Example 6.1, it is easy to check that the assumptions of the theorem are satisfied, and so we may apply it to get the following:

Corollary 6.4. *Let $p\in[2,\infty)$ and $q\in[p,2p]$. The following claims are equivalent:*

(i) *For all* $f \in L^p(\mathbb{R}^2)$ *there exists* $u \in \dot W^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$ *with* $\mathrm{J}u + \operatorname{div} u = f$.

(ii) *For all* $f \in L^p(\mathbb{R}^2)$ *there exists* $u \in \dot W^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$ *with*

$$\mathrm{J}u + \operatorname{div} u = f, \qquad \|\mathrm{D}u\|_{L^q}^q \le C\|f\|_{L^p}^p.$$

Remark 6.5. When $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, $n \le p < \infty$, $X = W_0^{1,p}(\Omega,\mathbb{R}^2)$ and $Y^* = L^p(\Omega)/\mathbb{R}$, the question about surjectivity of the operator $T = \operatorname{div} + \mathrm{J}$ is closely related to [34, Question 1.2]. It would be interesting to find out whether Theorems A and 6.2 can be adapted to bounded domains.

6.2 Examples

In this subsection we illustrate the use of Theorem 6.2 in the model cases of the 3D Navier-Stokes equations and Euler equation.

Example 6.6. We illustrate the use of Theorem 6.2 in the model case of the homogeneous, incompressible Navier-Stokes equations in $\mathbb{R}^3 \times [0,\infty)$:

$$\partial_t u + u\cdot\nabla u - \nu\Delta u - \nabla P = 0, \tag{6.7}$$

$$\operatorname{div} u = 0, \tag{6.8}$$

$$u(\cdot,0) = u^0, \tag{6.9}$$

where u is the velocity field, P is the pressure, $\nu > 0$ is the viscosity and u^0 is the initial data. The equations are invariant under the scalings $u \to u_\lambda$, $P \to P_\lambda$ and $u^0 \to u^0_\lambda$,

$$u_\lambda(x,t) \equiv \frac{1}{\lambda} u\left(\frac{x}{\lambda},\frac{t}{\lambda^2}\right), \qquad P_\lambda(x,t) \equiv \frac{1}{\lambda^2} P\left(\frac{x}{\lambda},\frac{t}{\lambda^2}\right), \qquad u^0_\lambda(x) = \frac{1}{\lambda}u^0\left(\frac{x}{\lambda}\right).$$

We divide the discussion into the following three steps: i) formally determining T; ii) choosing relevant ambient spaces X^* and Y^*; iii) choosing the domain of definition D.

We begin by choosing the operator T we wish to study. We incorporate (6.7)–(6.8) into the choice of the function spaces and choose, formally, $T(u) = u(\cdot,0)$. As the sought range we consider $Y^* = L^q_\sigma = \{v \in L^q(\mathbb{R}^3,\mathbb{R}^3)\colon \operatorname{div} v = 0\}$. We wish to choose the domain of definition D to be a suitable set of functions which satisfy (6.7)–(6.9) for some $u^0 \in L^q_\sigma$. We also need to determine the ambient space X^*.

In order for Theorem 6.2 to be applicable, we wish to consider regularity regimes where T is weakly* sequentially continuous and the sets $D_\ell = \{u \in D\colon \|u\|_X \le \ell,\ \|Tu\|_{Y^*} \le \ell\}$ are weakly* compact. It is natural to set $X^* = L^p_t(L^q_\sigma)_x \cap L^r_t \dot W^{1,s}_x$ for suitable $p,q,r,s \in [1,\infty]$. For condition $(\widehat{\mathrm{A2}})$ of Theorem 6.2 we compute, for all $p,q,r,s \in [1,\infty]$,

$$\begin{aligned}
\|u^0_\lambda\|_{L^q} &= \lambda^{3/q-1}\|u^0\|_{L^q},\\
\|u_\lambda\|_{L^p_tL^q_x} &= \lambda^{2/p+3/q-1}\|u\|_{L^p_tL^q_x},\\
\|u_\lambda\|_{L^r_t\dot W^{1,s}_x} &= \lambda^{2/r+3/s-2}\|u\|_{L^r_t\dot W^{1,s}_x}.
\end{aligned}$$

Thus $(\widehat{\mathrm{A2}})$ requires the compatibility condition $2/p+3/q-1 = 2/r+3/s-2$ to hold.

For simplicity, we set $q = 2$ and we consider the most familiar choice of exponents, that is, $X^* = L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot W^{1,2}_x$. Recall that $u \in X$ is called a *weak solution* of (6.7)–(6.9) if u satisfies

$$\int_0^\tau \langle u, \partial_t\varphi\rangle \,\mathrm{d}t + \int_0^\tau \langle u\otimes u, \mathrm{D}\varphi\rangle \,\mathrm{d}t - \nu\int_0^\tau \langle \mathrm{D}u, \mathrm{D}\varphi\rangle \,\mathrm{d}t + \langle u^0, \varphi(0)\rangle - \langle u(\tau), \varphi(\tau)\rangle = 0 \quad (6.10)$$

for all $\varphi \in C^\infty_c(\mathbb{R}^3\times[0,\infty), \mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$ and almost every $\tau > 0$. In (6.10), $\langle\cdot,\cdot\rangle$ denotes the inner product in L^2_x. This prompts us to set

$$D \equiv \{u \in L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot W^{1,2}_x : u \text{ is a weak solution of (6.7)–(6.9) for some } u^0 \in L^2_\sigma\},$$
$$T\colon D \to Y^*, \quad T(u) \equiv u^0 \text{ if (6.10) holds.}$$

We briefly indicate why $T\colon (D, \mathrm{wk}^*) \to (Y^*, \mathrm{wk}^*)$ is sequentially continuous and the sets D_ℓ are weakly* closed for all $\ell \in \mathbb{N}$. When $u \in D$, we have $\partial_t u \in L^{4/3}(0,\tau,(W^{1,2}_\sigma)^*)$ for all $\tau > 0$ (see [61, Lemma 3.7]). Thus, by using the Aubin–Lions lemma and a diagonal argument, if $u_j \overset{*}{\rightharpoonup} u$ in D, then every subsequence has a subsequence converging strongly in $L^2(0,T,L^2(B_R,\mathbb{R}^3))$ for all $T, R > 0$. The strong convergence and (6.10) imply that every subsequence of $(u^0_j)_{j\in\mathbb{N}}$ has a subsequence converging weakly* to u^0. This implies the two claims made above.

Theorem 6.2 now says that solvability of (6.7)–(6.9) for all $u^0 \in L^2_\sigma$ is equivalent to solvability with the *a priori* estimate

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^2_t \dot W^{1,2}_x} \le C\|u(\cdot,0)\|_{L^2}.$$

Such an estimate is satisfied by Leray–Hopf solutions [61].

Example 6.7. Our next aim is to prove Theorem E on energy dissipating solutions of the incompressible Euler equations in $\mathbb{R}^n\times[0,\infty)$, $n \ge 2$. Recall that given $u^0 \in L^2_\sigma$, a mapping $u \in L^p_t L^2_{\sigma,x}$, $2 \le p \le \infty$, is a weak solution of the Cauchy problem (1.7)–(1.9) if

$$\int_0^\infty \int_{\mathbb{R}^n} (u\cdot\partial_t\varphi + u\otimes u : \mathrm{D}\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} u^0\cdot\varphi(\cdot,0)\,\mathrm{d}t = 0 \quad \forall\varphi \in C^\infty_{c,\sigma}(\mathbb{R}^n\times[0,\infty),\mathbb{R}^n). \quad (6.11)$$

We cannot deduce Theorem E directly via Theorem 6.2. Indeed, the integral condition (6.11) leads to a well defined mapping T from a weak solution $u \in L^p_t L^2_{\sigma,x}$ of (1.7)–(1.9) to the initial data $u^0 \in L^2_\sigma$ but does not easily lend itself to a domain of definition $D \subset L^p_t L^2_{\sigma,x}$ satisfying condition $(\widehat{\mathrm{A4}})$ of Theorem 6.2. We therefore consider a relaxed problem where $u\otimes u \in L^{p/2}_t L^1_x$ is replaced by a general matrix-valued mapping S.

In order to apply Theorem 6.2 we embed $L^1(\mathbb{R}^n,\mathbb{R}^{n\times n})$ into the space of signed Radon measures $\mathbf{M}(\mathbb{R}^n,\mathbb{R}^{n\times n})$ which is the dual of the separable Banach space $C_0(\mathbb{R}^n,\mathbb{R}^{n\times n})$. We endow $\mathbf{M}(\mathbb{R}^n,\mathbb{R}^{n\times n})$ with the dual norm. In the relaxed problem we require $u \in L^p_t L^2_{\sigma,x}$ and $S \in L^{p/2}_t \mathbf{M}_x$ to satisfy

$$\int_0^\infty \int_{\mathbb{R}^n} (u\cdot\partial_t\varphi + S : \mathrm{D}\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} u^0\cdot\varphi(\cdot,0)\,\mathrm{d}t = 0 \quad \forall\varphi \in C^\infty_{c,\sigma}(\mathbb{R}^n\times[0,\infty),\mathbb{R}^n). \quad (6.12)$$

The point is that unlike (6.11), *condition* (6.12) *is stable under weak** *convergence because of linearity.* Theorem E is proved via the following lemma.

Lemma 6.8. *Let $n \geq 2$ and $p \in (2, \infty)$. For a residual set of data $u^0 \in L^2_\sigma$ there exists no couple $(u, S) \in L^p_t L^2_{\sigma,x} \times L^{p/2}_t \mathbf{M}_x$ satisfying* (6.12).

Proof. Assume, for the sake of contradiction, that for a non-meagre set of data $u^0 \in L^2_\sigma$ there exists a solution $(u, S) \in L^p_t L^2_{\sigma,x} \times L^{p/2}_t \mathbf{M}_x$ of (6.12).

Denote $D = \{(u, S) \in L^p_t L^2_{\sigma,x} \times L^{p/2}_t \mathbf{M}_x \colon \text{(6.12) holds for some } u^0 \in L^2_\sigma\}$. Let us define $T \colon D \to L^2_\sigma$ by $T(u, S) \equiv u^0$. Our intention is to verify the assumptions of Theorem 6.2 with two incompatible scalings. A simple application of Theorem 6.2 then implies the claim.

Let $(u, S) \in D$ and fix $\alpha, \beta \in \mathbb{R}$. Given $\lambda > 0$ we set

$$u_{\lambda,\alpha,\beta}(x,t) \equiv \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right), \tag{6.13}$$

$$S_{\lambda,\alpha,\beta}(x,t) \equiv \frac{1}{\lambda^{2\alpha}} S\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right), \tag{6.14}$$

$$u^0_{\lambda,\alpha,\beta}(x,t) \equiv \frac{1}{\lambda^\alpha} u^0\left(\frac{x}{\lambda^\beta}\right). \tag{6.15}$$

Now (6.12) and (6.13)–(6.15) imply that

$$\int_0^\infty \int_{\mathbb{R}^n} (u_{\lambda,\alpha,\beta} \cdot \partial_t \varphi + S_{\lambda,\alpha,\beta} : \mathrm{D}\varphi) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^n} u^0_{\lambda,\alpha,\beta} \cdot \varphi(\cdot, 0) \,\mathrm{d}t = 0$$

for all $\varphi \in C^\infty_{c,\sigma}(\mathbb{R}^n \times [0,\infty); \mathbb{R}^n)$ so that $(u_{\lambda,\alpha,\beta}, S_{\lambda,\alpha,\beta}) \in D$ and $T(u_{\lambda,\alpha,\beta}, S_{\lambda,\alpha,\beta}) = u^0_{\lambda,\alpha,\beta}$.

Now

$$\begin{aligned}
\|u_{\lambda,\alpha,\beta}\|_{L^p_t L^2_x} &= \lambda^{\frac{n\beta}{2} + \frac{\alpha+\beta}{p} - \alpha} \|u\|_{L^p_t L^2_x} = \lambda^{\frac{(np+2)\beta - (2p-2)\alpha}{2p}} \|u\|_{L^p_t L^2_x}, \\
\|S_{\lambda,\alpha,\beta}\|_{L^{p/2}_t \mathbf{M}_x} &= \lambda^{n\beta + \frac{2(\alpha+\beta)}{p} - 2\alpha} \|S\|_{L^{p/2}_t \mathbf{M}_x} = \lambda^{\frac{(np+2)\beta - (2p-2)\alpha}{p}} \|S\|_{L^{p/2}_t \mathbf{M}_x}, \\
\|u^0_{\lambda,\alpha,\beta}\|_{L^2} &= \lambda^{\frac{n\beta}{2} - \alpha} \|u^0\|_{L^2} = \lambda^{\frac{n\beta - 2\alpha}{2}} \|u^0\|_{L^2}.
\end{aligned}$$

As long as the powers of λ above are positive, Theorem 6.2 implies that for every $u^0 \in L^2_\sigma$ we can choose a solution of (6.12) such that it satisfies

$$\|u\|_{L^p_t L^2_x}^{\frac{np\beta - 2p\alpha}{(np+2)\beta - (2p-2)\alpha}} + \|S\|_{L^{p/2}_t \mathbf{M}_x}^{\frac{np\beta/2 - p\alpha}{(np+2)\beta - (2p-2)\alpha}} \leq C_{\alpha,\beta} \|u^0\|_{L^2}.$$

Let now $0 < \varepsilon < np/(np+2)$. Choose $\alpha, \beta > 0$ such that

$$\frac{np\beta - 2p\alpha}{(np+2)\beta - (2p-2)\alpha} = \varepsilon.$$

We conclude that for every $u^0 \in L^2_\sigma$ there exists a solution (u, S) of (6.12) with

$$\|u\|^\varepsilon_{L^p_t L^2_x} + \|S\|^{\frac{\varepsilon}{2}}_{L^{p/2}_t \mathbf{M}_x} \leq C_\varepsilon \|u^0\|_{L^2}.$$

Let $\|u^0\|_{L^2} = 1$ and fix $\gamma > 0$ with $\beta - \gamma$ small. Choose $(\tilde{u}, \tilde{S})$ such that

$$T(\tilde{u}, \tilde{S}) = u^0_{\lambda,\alpha,\gamma}, \qquad \|\tilde{u}\|^\varepsilon_{L^p_T L^2_x} + \|\tilde{S}\|^{\varepsilon/2}_{L^{p/2}_t \mathbf{M}_x} \leq C_\varepsilon \|u^0_{\lambda,\alpha,\gamma}\|_{L^2} = C_\varepsilon \lambda^{\frac{n\gamma - 2\alpha}{2}}.$$

Next choose $(u,S) \in L^p_t L^2_{\sigma,x} \times L^{p/2}_t \mathbf{M}_x$ such that $(u_{\lambda,\alpha,\beta}, S_{\lambda,\alpha,\beta}) = (\tilde{u}, \tilde{S})$. Now $T(u,S) = u^0$ but

$$\begin{aligned}\|u\|_{L^p_t L^2_x} &= \lambda^{-\frac{(np+2)\beta-(2p-2)\alpha}{2p}} \|\tilde{u}\|_{L^p_t L^2_x} = \lambda^{-\frac{n\beta-2\alpha}{2\varepsilon}} \|\tilde{u}\|_{L^p_t L^2_x} \\ &\leq C_\varepsilon^{\frac{1}{\varepsilon}} \lambda^{\frac{n\gamma-2\alpha}{2\varepsilon} - \frac{n\beta-2\alpha}{2\varepsilon}} = C_\varepsilon^{\frac{1}{\varepsilon}} \lambda^{\frac{n(\gamma-\beta)}{2\varepsilon}}.\end{aligned}$$

By varying $\lambda > 0$ we get $u = 0$, which yields a contradiction with $u(\cdot, 0) = u^0$. □

It is intriguing that despite being highly underdetermined, (6.12) generically requires that the kinetic energy of u does not enjoy L^q decay. Note, for instance, that for every $u^0 \in L^2_\sigma$ one gets a solution of (6.12) by setting $u(x,t) \equiv u^0(x)$ and $S(x,t) \equiv 0$.

Proof of Theorem E. Let $M > 0$ and suppose, by way of contradiction, that the set of data with a solution $u \in M\mathbb{B}_{L^p_t L^2_{\sigma,x}}$ is not nowhere dense in L^2_σ. In particular, defining T as in the proof of Lemma 6.8, $T(\{(u,S) \in D \colon (u,S) \in M\mathbb{B}_{L^p_t L^2_{\sigma,x}} \times M\mathbb{B}_{L^{p/2}_t \mathbf{M}_x}\})$ is not nowhere dense in L^2_σ. On the other hand, $T(\{(u,S) \in D \colon (u,S) \in M\mathbb{B}_{L^p_t L^2_{\sigma,x}} \times M\mathbb{B}_{L^{p/2}_t \mathbf{M}_x}\})$ is closed in L^2_σ, and therefore it has a non-empty interior, contradicting Lemma 6.8. □

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